Eur. Phys. J. C 20, 723–757 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100694

THE EUROPEAN
PHYSICAL JOURNAL C

On the equivalence between sine-Gordon model and Thirring model in the chirally broken phase of the Thirring model

M. Faber[a], A.N. Ivanov[b,c]

Atominstitut der Österreichischen Universitäten, Arbeitsbereich Kernphysik und Nukleare Astrophysik, Technische Universität Wien, Wiedner Hauptstrasse 8–10, 1040 Wien, Österreich

Received: 16 December 2000 / Revised version: 23 April 2001 /
Published online: 13 June 2001 –

Abstract. We investigate the equivalence between Thirring model and sine-Gordon model in the chirally broken phase of the Thirring model. This is unlike all other available approaches where the fermion fields of the Thirring model were quantized in the chiral symmetric phase. In the path integral approach we show that the bosonized version of the massless Thirring model is described by a quantum field theory of a massless scalar field and exactly solvable, and the massive Thirring model bosonizes to the sine-Gordon model with a new relation between the coupling constants. We show that the non-perturbative vacuum of the chirally broken phase in the massless Thirring model can be described in complete analogy with the BCS ground state of superconductivity. The Mermin–Wagner theorem and Coleman's statement concerning the absence of Goldstone bosons in the $1+1$-dimensional quantum field theories are discussed. We investigate the current algebra in the massless Thirring model and give a new value of the Schwinger term. We show that the topological current in the sine-Gordon model coincides with the Noether current responsible for the conservation of the fermion number in the Thirring model. This allows one to identify the topological charge in the sine-Gordon model with the fermion number.

1 Introduction

In $1+1$-dimensional space-time there are two non-trivial minimal quantum field theories which describe non-perturbative phenomena: the sine-Gordon (SG) model [1] and the Thirring model [2]. The SG model is a quantum field theory of a single scalar field $\vartheta(x)$ self-coupled through the dynamics determined by the Lagrangian [3][1]

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1), \quad (1.1)$$

where α and β are real positive parameters [3]. The Lagrangian (1.1) is invariant under the transformations

$$\vartheta(x) \to \vartheta'(x) = \vartheta(x) + \frac{2\pi n}{\beta}, \quad (1.2)$$

where n is an integer number running over $n = 0, \pm 1, \pm 2, \ldots$

The most interesting property of the SG model is the existence of classical, stable solutions of the equations of motion – solitons and anti-solitons [1]. Solitons can annihilate with anti-solitons. Many-soliton solutions obey Pauli's exclusion principle. As pointed out by Skyrme [4] this can be interpreted as a fermion-like behavior.

In turn, the Thirring model [2] is a theory of a self-coupled Dirac field $\psi(x)$ [2,3]

$$\begin{aligned}\mathcal{L}(x) &= \bar{\psi}(x)(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x)\\ &\quad - \frac{1}{2}g\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x)\gamma_\mu\psi(x),\end{aligned} \quad (1.3)$$

where m is the mass of the fermion field and g is a dimensionless coupling constant. The field $\psi(x)$ is a spinor field with two components $\psi_1(x)$ and $\psi_2(x)$. The γ matrices are defined in terms of the well-known 2×2 Pauli matrices σ_1, σ_2 and σ_3

$$\gamma^0 = \sigma_1, \quad \gamma^1 = -\mathrm{i}\sigma_2, \quad \gamma^5 = \gamma^0\gamma^1 = -\mathrm{i}\sigma_1\sigma_2 = \sigma_3 \quad (1.4)$$

These γ matrices obey the standard relations

$$\begin{aligned}&\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},\\ &\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0.\end{aligned} \quad (1.5)$$

We use the metric tensor $g^{\mu\nu}$ defined by $g^{00} = -g^{11} = 1$ and $g^{01} = g^{10} = 0$. The axial-vector product $\gamma^\mu\gamma^5$ can be expressed in terms of γ^ν

$$\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu, \quad (1.6)$$

[a] e-mail: faber@kph.tuwien.ac.at
[b] e-mail: ivanov@kph.tuwien.ac.at
[c] *Permanent address:* State Technical University, Department of Nuclear Physics, 195251 St. Petersburg, Russian Federation
[1] Below we follow Coleman's notation [3]

where $\epsilon^{\mu\nu}$ is the anti-symmetric tensor defined by $\epsilon^{01} = -\epsilon^{10} = 1$. Further, we also use the relation $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma^5$.

The Lagrangian (1.3) is obviously invariant under $U_{\rm V}(1)$ transformations

$$\psi(x) \stackrel{\rm V}{\longrightarrow} \psi'(x) = e^{{\rm i}\alpha_{\rm V}}\psi(x). \tag{1.7}$$

For $m = 0$ the Lagrangian (1.3) is invariant under the chiral group $U_{\rm V}(1) \times U_{\rm A}(1)$

$$\begin{aligned} \psi(x) &\stackrel{\rm V}{\longrightarrow} \psi'(x) = e^{{\rm i}\alpha_{\rm V}}\psi(x), \\ \psi(x) &\stackrel{\rm A}{\longrightarrow} \psi'(x) = {\rm e}^{{\rm i}\alpha_{\rm A}\gamma^5}\psi(x), \end{aligned} \tag{1.8}$$

where $\alpha_{\rm V}$ and $\alpha_{\rm A}$ are real parameters defining global rotations.

As has been shown in [5,6] the massless Thirring model can be exactly solved in the sense that all correlation functions can be calculated explicitly. The solution of the massless Thirring model has been obtained in the traditional quantum field theoretic way by Klaiber [5] within the operator technique and within the path integral approach by Furuya, Gamboa Saravi and Schaposnik [6] by using the technique of auxiliary vector fields. Within the path integral approach and without the introduction of auxiliary vector fields Fröhlich and Marchetti [7] have shown that the evaluation of the Green functions in the massless Thirring model runs parallel to the evaluation of the Green functions in the quantum field theory of a massless scalar field coupled to external sources via SG model-like couplings.

The problem of the equivalence between the SG and the Thirring model has a long history. The first discussion of this topic has been started by Skyrme [4] and continued by Coleman [3] and Mandelstam [8]. Skyrme argued that the soliton modes of the SG model possess the properties of fermion fields and couple through an interaction of Thirring model type. Coleman suggested a perturbative approach to the understanding of the equivalence between the SG and the Thirring model. He developed a perturbation theory with respect to α and m in order to compare the n-point Green functions in the SG and the massive Thirring model in coordinate representation. Under the assumption of the existence of these two theories in the strict sense of constructive quantum field theory, Coleman concluded that they should be equivalent if the coupling constants β and g obey the relation [3]

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi} \tag{1.9}$$

and the operators $\psi(x)$ and $\vartheta(x)$ satisfy the Abelian bosonization rules [3]

$$Zm\bar{\psi}(x)\left(\frac{1 \mp \gamma^5}{2}\right)\psi(x) = -\frac{\alpha}{\beta^2}{\rm e}^{\pm{\rm i}\beta\vartheta(x)}, \tag{1.10}$$

where the constant Z depends on the regularization [3]. The results obtained by Coleman are fully based on the solution of the massless Thirring model given by Klaiber [5] and recovered in a pure Euclidean formulation by Fröhlich and Marchetti [7].

Unlike Coleman's analysis dealing with Green functions, i.e. matrix elements of some products of field operators, Mandelstam has undertaken an attempt of an explicit derivation of the operators being functionals of the scalar field of the SG model and possessing the properties of the fermionic field operators. Mandelstam identified these operators with the interpolating operators of Thirring fields and showed that these fermionic operators have a Lagrangian of the Thirring model type.

Recently, another approach to the derivation of the equivalence between the SG and the Thirring model was developed by Damgaard, Nielsen and Sollacher [9] and Thomassen [10] within the so-called *smooth bosonization approach* based on the path integral method and using an enlarged set of field variables. In the *smooth bosonization approach* this enlargement of field variables appears via local chiral rotations [11], where the local chiral phase is identified with a local pseudoscalar field. Its Lagrangian is determined by the Jacobian of the fermion path integral measure depending explicitly on a local chiral phase [12–17].

The common point of all approaches to the solution of the massless Thirring model [5–7] and to the derivation of the equivalence between the SG and the massive Thirring model [3,9,10] is a quantization of the fermionic system around the trivial perturbative vacuum.

In order to justify our statement we suggest to follow the procedure used by Nambu and Jona–Lasinio [18]. Let us consider the massless Thirring model defined by the Lagrangian (1.3) at $m = 0$. Then, following Nambu and Jona–Lasinio we supplement and subtract the term $M\bar{\psi}(x)\psi(x)$, where M is an arbitrary parameter with the meaning of the dynamical mass of fermions. That is similar to the Hartee–Fock approximation where a two-body interaction is approximated by a one-body term. The Lagrangian of the massless Thirring model acquires the form

$$\mathcal{L}(x) = \bar{\psi}(x)({\rm i}\gamma^\mu\partial_\mu - M)\psi(x) + \mathcal{L}_{\rm int}(x), \tag{1.11}$$

with the interaction $\mathcal{L}_{\rm int}(x)$ given by

$$\begin{aligned} \mathcal{L}_{\rm int}(x) &= M\bar{\psi}(x)\psi(x) \\ &\quad -\frac{1}{2}g\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x)\gamma_\mu\psi(x). \end{aligned} \tag{1.12}$$

Following the Nambu–Jona–Lasinio prescription one can show that the dynamical mass M satisfies the gap equation

$$\begin{aligned} M &= g\gamma^\mu(-{\rm i})S_{\rm F}(0)\gamma_\mu \\ &= g\gamma^\mu\int\frac{{\rm d}^2p}{(2\pi)^2{\rm i}}\frac{1}{M - \hat{p}}\gamma_\mu \\ &= M\frac{g}{2\pi}\ln\left(1 + \frac{\Lambda^2}{M^2}\right), \end{aligned} \tag{1.13}$$

where Λ is an ultra-violet cut-off. Thus the gap equation reads[2]

$$M = M\frac{g}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right). \qquad (1.14)$$

There are two solutions of this equation: $M = 0$ and

$$M = \frac{\Lambda}{\sqrt{e^{2\pi/g}-1}}. \qquad (1.15)$$

The $M = 0$ solution is trivial and corresponds to a chiral symmetric phase with a trivial perturbative vacuum. In turn, the $M \neq 0$ solution (1.15) is non-trivial and is related to the chirally broken phase with a non-perturbative vacuum. This chirally broken phase is characterized by the appearance of dynamical fermions with a dynamical mass M and $\bar{\psi}\psi$ pairing [18]. By retracing [3,5–11] it becomes obvious that all results obtained there can be assigned to the $M = 0$ solution characterizing the quantization of fermion fields around the trivial perturbative vacuum.

In order to show that the chirally broken $M \neq 0$ phase of the Thirring model is more preferable than the chiral symmetric $M = 0$ phase we have to calculate the energy density of the vacuum state $\mathcal{E}(M)$. This can be carried out only by using the exact expression for the wave function of the non-perturbative vacuum. In Sect. 6 we show that the wave function of the non-perturbative vacuum in the massless Thirring model can be taken in the form of the wave function of the ground state in the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity [23] (see also [18,22,24]). The energy density $\mathcal{E}(M)$ calculated in Sect. 6 has a minimum at $M \neq 0$ that satisfies the gap equation (1.14) and a maximum at $M = 0$. This is evidence that for the Thirring fermions the chirally broken phase is energetically preferable with respect to the chiral symmetric phase.

It is well known that the chirally broken phase is characterized by a non-zero value of the fermion condensate, $\langle 0|\bar{\psi}(0)\psi(0)|0\rangle \neq 0$. In the massless Thirring model the fermion condensate is defined by

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\rm one\ loop} = {\rm i}\,{\rm tr}\{S_{\rm F}(0)\} = -\frac{M}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right) = -\frac{M}{g}, \qquad (1.16)$$

where we have taken into account the gap equation (1.14).

Below we denote the fermion condensate (1.16) calculated in the one-fermion loop approximation by $\langle\bar{\psi}\psi\rangle$, $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\rm one\ loop} = \langle\bar{\psi}\psi\rangle = -M/g$.

Since the massless Thirring model possesses the same non-perturbative properties as the Nambu–Jona–Lasinio model [18], we suggest to recast the four-fermion interaction of the Thirring model into the form given by Nambu and Jona–Lasinio. After a Fierz transformation

$$-\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x)\gamma_\mu\psi(x) = (\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x){\rm i}\gamma^5\psi(x))^2, \qquad (1.17)$$

the Lagrangian (1.3) acquires the form

$$\mathcal{L}(x) = \bar{\psi}(x)({\rm i}\gamma^\mu\partial_\mu - m)\psi(x) + \frac{1}{2}g\left[(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x){\rm i}\gamma^5\psi(x))^2\right]. \qquad (1.18)$$

In this form the Thirring model coincides with the Nambu–Jona–Lasinio (NJL) model in $1+1$-dimensional space-time. It is well known that the NJL model is a relativistic covariant generalization of the BCS theory of superconductivity. The wave function of the non-perturbative vacuum of the NJL model coincides with the wave function of the ground state in the BCS theory [23].

The main aim of this article is to solve the massless Thirring model in the chirally broken phase and to derive the equivalence with the SG model. We also want to show explicitly that this is possible without an enlargement of the number of degrees of freedom but via a reduction of them. In fact, in the Thirring model the fermion field has two independent degrees of freedom. Since the SG model describes a scalar field with only one degree of freedom, one of the two fermion degrees of freedom should die out. How this goes dynamically in a non-perturbative way should be the matter of our investigation.

In this way it is rather useful to follow the approach developed in [19–22] for the derivation of effective chiral Lagrangians in the extended Nambu–Jona–Lasinio (ENJL) model with chiral $U(3)\times U(3)$ symmetry [19–21] and the effective Lagrangian in the monopole Nambu–Jona–Lasinio model with magnetic $U(1)$ symmetry [22].

This paper is organized as follows. In Sect. 2 we discuss Coleman's derivation of the equivalence between the massive Thirring model and the SG model. In Sect. 3 we bosonize the massless Thirring model. We show that the bosonized version of the massless Thirring model is a quantum field theory of a free massless scalar field. In Sect. 4 we evaluate the generating functional of Green functions in the massless Thirring model and show that any Green function in the massless Thirring model can be expressed in terms of vacuum expectation values of the operators $e^{\pm {\rm i}\beta\vartheta(x)}$, where $\vartheta(x)$ is a massless scalar field with values $0 \le \vartheta(x) \le 2\pi$. Using a trivial cut-off regularization of the massless ϑ field in the infrared region we obtain results that coincide with those derived by Klaiber, Coleman, Fröhlich and Marchetti, but give another relation between the coupling constant β and g than that given by Coleman [3]. The problem of the vanishing of the fermion condensate averaged over the ϑ field fluctuations is accentuated. The solution of this problem is discussed in Sect. 8. In Sect. 5 we bosonize the massive Thirring model. We show that the bosonized version of the massive Thirring model is just the SG model. We express the parameters of the SG model in terms of the parameters of the massive Thirring model. In Sect. 6 we investigate the massless

[2] We would like to accentuate that the gap equation is calculated in the one-fermion loop approximation. As has been shown in [19–22] the effective Lagrangian of a bosonized version of a fermion system self-coupled via a four-fermion interaction is defined by a functional determinant that can be represented in terms of an infinite series of one-fermion loop diagrams

Thirring model in the operator formulation. We analyze the normal ordering of the fermionic operators and chiral symmetry breaking, the equations of motion for fermion fields, the current algebra and the energy-momentum tensor. We discuss the phenomenon of spontaneous breaking of chiral symmetry in the massless Thirring model from the point of view of the BCS theory of superconductivity. We use the exact expression for the wave function of the non-perturbative vacuum and calculate the energy density of this non-perturbative vacuum state. We show that the energy density of the non-perturbative vacuum acquires a minimum just, when the dynamical mass M of fermions satisfies the gap equation (1.14). Then, we show that the Schwinger term calculated for the fermion system in the chirally broken phase becomes depending on the coupling constant g. In Sect. 7 we show that the topological current of the SG model coincides with the Noether current of the massive Thirring model related to the invariance under global $U_{\rm V}(1)$ rotation. As far as this Noether current is responsible for the conservation of the fermion number in the massive Thirring model, the topological charge of soliton solutions of the SG model inherits the meaning of the fermion number. This proves Skyrme's statement [4] that the SG model solitons can be interpreted as massive fermions. In Sect. 8 we discuss the spontaneous breaking of chiral symmetry in the massless Thirring model, the Mermin–Wagner theorem [25] about the vanishing of long-range order in two-dimensional quantum field theories and Coleman's statement concerning the absence of Goldstone bosons in the $1+1$-dimensional quantum field theory of a massless scalar field. We show that in our approach the problem of the vanishing of the fermion condensate averaged over the ϑ field fluctuations can be solved by means of dimensional and analytical regularization. We give the solution of the massless Thirring model in the sense of an explicit evaluation of any correlation function. In the Conclusion we discuss our results. In Appendix A we calculate the Jacobian caused by local chiral rotations and show that by using an appropriate regularization scheme this Jacobian can be equal to unity. In Appendix B we demonstrate the stability of the chirally broken phase under fluctuations of the radial scalar field around the minimum of the effective potential calculated in Sect. 3. We show that the radial scalar field fluctuating around the minimum of the effective potential is decoupled from the system. In Appendix C we give a classical solution of the equations of motion of the massless Thirring model for the ansatz discussed in Sect. 6. In Appendix D we give a detailed description of free massive and massless fermion fields in $1+1$-dimensional space-time.

2 On Coleman's analysis of equivalence

In this section we would like to repeat Coleman's derivation of the equivalence between the massive Thirring model and the SG model within the path integral approach. Let $Z_{\rm SG}$ and $Z_{\rm Th}$ be the partition functions of the SG and the massive Thirring model defined by

$$\begin{aligned}
Z_{\rm SG} &= \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x \left\{\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1)\right\},\\
Z_{\rm Th} &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\\
&\quad\times \exp{\rm i}\int {\rm d}^2x\left\{\bar{\psi}(x)({\rm i}\gamma^\mu\partial_\mu - m)\psi(x) - \frac{1}{2}g\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(x)\gamma^\mu\psi(x)\right\}.
\end{aligned} \tag{2.1}$$

Formally, in order to get convinced that the SG and the massive Thirring model are equivalent it is sufficient to show that $Z_{\rm SG} = Z_{\rm Th}$. Coleman suggested to prove this relation perturbatively. For this aim he developed perturbation theories with respect to α and m [3]. According to Coleman we have to expand the partition functions with respect to the interaction terms [3]:

$$\begin{aligned}
\mathcal{L}^{\rm SG}_{\rm int}(x) &= \frac{\alpha}{\beta^2}\cos\beta\vartheta(x) = \frac{\alpha}{2\beta^2}(A_+(x) + A_-(x)),\\
\mathcal{L}^{\rm Th}_{\rm int}(x) &= -m\bar{\psi}(x)\psi(x) = -m(\sigma_+(x) + \sigma_-(x)),
\end{aligned} \tag{2.2}$$

where [3]

$$\begin{aligned}
A_\pm(x) &= {\rm e}^{\pm{\rm i}\beta\vartheta(x)},\\
\sigma_\pm(x) &= \bar{\psi}(x)\left(\frac{1\pm\gamma^5}{2}\right)\psi(x).
\end{aligned} \tag{2.3}$$

In terms of the components $\psi_1(x)$ and $\psi_2(x)$ the fermion densities $\sigma_\pm(x)$ are defined by $\sigma_+(x) = \psi^\dagger_2(x)\psi_1(x)$ and $\sigma_-(x) = \psi^\dagger_1(x)\psi_2(x)$.

The expansions for the partition functions in powers of α and m read

$$\begin{aligned}
Z_{\rm SG} &= \sum^\infty_{n=0}\frac{{\rm i}^n}{n!}\left(\frac{\alpha}{2\beta^2}\right)^n\int\mathcal{D}\vartheta\exp{\rm i}\int{\rm d}^2x\\
&\quad\times\left\{\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x)\right\}\int\!\!\int\ldots\int{\rm d}^2x_1{\rm d}^2x_2\ldots{\rm d}^2x_n\\
&\quad\times\prod^n_{k=1}(A_+(x_k) + A_-(x_k)),\\
Z_{\rm Th} &= \sum^\infty_{n=0}\frac{{\rm i}^n}{n!}(-m)^n\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp{\rm i}\int{\rm d}^2x\\
&\quad\times\left\{\bar{\psi}(x){\rm i}\gamma^\mu\partial_\mu\psi(x) - \frac{1}{2}g\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(x)\gamma^\mu\psi(x)\right\}\\
&\quad\times\int\!\!\int\ldots\int{\rm d}^2x_1{\rm d}^2x_2\ldots{\rm d}^2x_n\\
&\quad\times\prod^n_{k=1}(\sigma_+(x_k) + \sigma_-(x_k)).
\end{aligned} \tag{2.4}$$

Every term of these expansions corresponds to a vacuum expectation value of a massless free scalar field $\vartheta(x)$[3] and a massless self-coupled fermion field $\psi(x)$.

A general term of $Z_{\rm SG}$ can be taken in the form [3]

$$\left\langle 0\left|T\left(\prod_{k=1}^{p}A_+(x_k)\prod_{j=1}^{n}A_-(y_j)\right)\right|0\right\rangle = \int \mathcal{D}\vartheta {\rm e}^{-{\rm i}(1/2)\int {\rm d}^2x\vartheta(x)(\Box+\mu^2)\vartheta(x)} \times \prod_{k=1}^{p}A_+(x_k)\prod_{j=1}^{n}A_-(y_j), \qquad (2.5)$$

where μ is an infrared cut-off regularizing the free massless ϑ field in the infrared region. The vacuum expectation value (2.5) should be taken in the limit $\mu\to 0$ [3].

The evaluation of this Gaussian path integral is rather straightforward. The result reads

$$\left\langle 0\left|T\left(\prod_{i=1}^{p}A_+(x_i)\prod_{j=1}^{n}A_-(y_j)\right)\right|0\right\rangle = \exp\left\{\frac{1}{2}\beta^2(p+n){\rm i}\Delta(0)\right\}\exp\left\{\beta^2\sum_{j<k}^{p}{\rm i}\Delta(x_j-x_k) +\beta^2\sum_{j<k}^{n}{\rm i}\Delta(y_j-y_k)-\beta^2\sum_{j=1}^{p}\sum_{k=1}^{n}{\rm i}\Delta(x_j-y_k)\right\}, \qquad (2.6)$$

where the Green function $\Delta(x-y)$ is determined by [3]

$$\Delta(x-y)={\rm i}\langle 0|T(\vartheta(x)\vartheta(y))|0\rangle \qquad (2.7)$$

and obeys the equation [3]

$$(\Box+\mu^2)\Delta(x-y)=\delta^{(2)}(x-y). \qquad (2.8)$$

In the limit $\mu\to 0$ the Green function $\Delta(x-y)$ is given by the expression [3]

$$\Delta(x-y)=-\frac{\rm i}{4\pi}\ln[-\mu^2(x-y)^2]. \qquad (2.9)$$

Using the explicit form of the Green function (2.9) the vacuum expectation value (2.6) transforms to

$$\left\langle 0\left|T(\prod_{k=1}^{p}A_+(x_k)\prod_{j=1}^{n}A_-(y_j))\right|0\right\rangle = \exp\left\{\frac{1}{2}\beta^2(p+n){\rm i}\Delta(0)\right\} \qquad (2.10)$$
$$\times\frac{\prod_{j<k}^{p}[-\mu^2(x_j-x_k)^2]^{\beta^2/4\pi}\prod_{j<k}^{n}[-\mu^2(y_j-y_k)^2]^{\beta^2/4\pi}}{\prod_{j=1}^{p}\prod_{k=1}^{n}[-\mu^2(x_j-y_k)^2]^{\beta^2/4\pi}},$$

in agreement with Coleman's result (see (4.11) of [3]).

In the limit $\mu^2\to 0$ this vacuum expectation value behaves like

$$\left\langle 0\left|T\left(\prod_{k=1}^{p}A_+(x_k)\prod_{j=1}^{n}A_-(y_j)\right)\right|0\right\rangle \sim (\mu^2)^{(p-n)^2\beta^2/8\pi} \qquad (2.11)$$

and vanishes if $p\neq n$ [3]. An analogous evaluation of the vacuum expectation value (2.5) has been carried out by Fröhlich and Marchetti [7].

We would like to accentuate that the evaluation of the vacuum expectation value (2.10) has been carried out with respect to the trivial perturbative vacuum with $\langle\vartheta(x)\rangle=0$ and with the trivial two-point Green function (2.9). Therefore, no non-perturbative properties of the SG model caused by the existence of non-trivial soliton states are involved.

Now let us turn to the evaluation of the partition function $Z_{\rm Th}$. From (2.4) one can see that every term of the expansion in powers of m is related to the vacuum expectation value of a product of operators of massless self-coupled fermion fields $\psi(x)$ and $\bar{\psi}(x)$

$$\left\langle 0\left|T\left(\prod_{k=1}^{n}\sigma_-(x_k)\sigma_+(y_k)\right)\right|0\right\rangle = \left\langle 0\left|T\left(\prod_{i=1}^{n}[\psi_1^{\dagger}(x_i)\psi_2(x_i)][\psi_2^{\dagger}(y_i)\psi_1(y_i)]\right)\right|0\right\rangle. \qquad (2.12)$$

For the evaluation of these vacuum expectation values Coleman has decided to follow as close as possible the results obtained by Klaiber in his lectures [5]. According to Klaiber's statement the massless Thirring model can be reduced to a quantum field theory of a massless free fermion field $\Psi(x)$ by a corresponding canonical transformation of the self-coupled fermion field $\psi(x)\to\Psi(x)$ [5]. The two-point Green function $S_{\rm F}(x-y)$ of the free massless field $\Psi(x)$ is defined by [5]

$$S_{\rm F}(x-y)={\rm i}\langle 0|T(\Psi(x)\bar{\Psi}(y))|0\rangle=\frac{1}{2\pi}\frac{\hat{x}-\hat{y}}{(x-y)^2}, \qquad (2.13)$$

where $\hat{x}-\hat{y}=\gamma^{\mu}(x-y)_{\mu}$. We would like to emphasize that the fermion condensate, determined in the usual way, is equal to zero:

$$\lim_{y\to x}{\rm i}\langle 0|T(\bar{\Psi}(y)\Psi(x))|0\rangle=-\lim_{y\to x}{\rm tr}[S_{\rm F}(x-y)] = 0. \qquad (2.14)$$

This shows clearly that fermion fields are quantized around the trivial perturbative vacuum.

The result of the calculation of the vacuum expectation value (2.12) was obtained by Klaiber [5] and used by Coleman [3] in the form

$$\left\langle 0\left|T\left(\prod_{k=1}^{n}\sigma_-(x_k)\sigma_+(y_k)\right)\right|0\right\rangle \qquad (2.15)$$

[3] Of course, in reality the ϑ field is a massless pseudoscalar field. As we show below (see also [3,6–16]) it is related to a chiral phase of a fermion field. Since we will not use the properties of the ϑ field under parity transformations, further on we call it for simplicity *a massless scalar field*

$$\sim \frac{\displaystyle\prod_{j<k}^{p}[-\bar{\mu}^2(x_j-x_k)^2]^{1+b/\pi}\prod_{j<k}^{n}[-\bar{\mu}^2(y_j-y_k)^2]^{1+b/\pi}}{\displaystyle\prod_{j=1}^{n}\prod_{k=1}^{n}[-\bar{\mu}^2(x_j-y_k)^2]^{1+b/\pi}},$$

where $\bar{\mu}$ is an arbitrary scale and the parameter b is given by [3,5]

$$1+\frac{b}{\pi}=\frac{1}{1+\dfrac{g}{\pi}}. \qquad (2.16)$$

The comparison of (2.10) for $p=n$ with (2.15) led Coleman to the relation between the coupling constants given by (1.9) at $\bar{\mu}\sim\mu$ [3].

The evaluation of the Green functions in the massless Thirring model carried out by Klaiber within the operator technique was then confirmed by Furuya, Gamboa Saravi and Schaposnik within the path integral approach supplemented by the method of auxiliary vector fields [6].

Thus, we have to emphasize that the fermion fields in Coleman's derivation of the equivalence between the SG and the Thirring model have been obviously quantized around the trivial perturbative vacuum in the chiral symmetric phase. Therefore, it is not a surprise that Coleman's relation between coupling constants differs from our relation valid for fermion fields quantized around a non-trivial, non-perturbative vacuum in the chirally broken phase.

3 Bosonization of the massless Thirring model

Within our approach to the equivalence between the SG and the Thirring model we suggest to start, first, with the massless Thirring model and bosonize it by integrating over fermionic degrees of freedom. We consider the partition function

$$Z_{\rm Th}=\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp{\rm i}\int{\rm d}^2x\Big\{\bar{\psi}(x){\rm i}\gamma^{\mu}\partial_{\mu}\psi(x) -\frac{1}{2}g\bar{\psi}(x)\gamma_{\mu}\psi(x)\bar{\psi}(x)\gamma^{\mu}\psi(x)\Big\}. \qquad (3.1)$$

After a Fierz transformation of the four-fermion interaction we get

$$Z_{\rm Th}=\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp{\rm i}\int{\rm d}^2x\Big\{\bar{\psi}(x){\rm i}\gamma^{\mu}\partial_{\mu}\psi(x) +\frac{1}{2}g[(\bar{\psi}(x)\psi(x))^2+(\bar{\psi}(x){\rm i}\gamma^5\psi(x))^2]\Big\}. \qquad (3.2)$$

Collective $\bar{\psi}\psi$ excitations, a local scalar field $\sigma(x)$ and a pseudoscalar $\varphi(x)$ can be introduced in the theory as usual [18–22]

$$Z_{\rm Th}=\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\mathcal{D}\varphi \times\exp{\rm i}\int{\rm d}^2x\Big\{\bar{\psi}(x){\rm i}\gamma^{\mu}\partial_{\mu}\psi(x)-\bar{\psi}(x)(\sigma(x) +{\rm i}\gamma^5\varphi(x))\psi(x)-\frac{1}{2g}[\sigma^2(x)+\varphi^2(x)]\Big\}. \qquad (3.3)$$

Integrating over fermionic degrees of freedom we recast the integrand into the form

$$Z_{\rm Th}=\int\mathcal{D}\sigma\mathcal{D}\varphi{\rm Det}({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi) \times\exp{\rm i}\int{\rm d}^2x\left\{-\frac{1}{2g}[\sigma^2(x)+\varphi^2(x)]\right\}. \qquad (3.4)$$

This reduces the problem of the bosonization of the massless Thirring model to the evaluation of the functional determinant

$${\rm Det}({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi). \qquad (3.5)$$

This determinant is related to the effective Lagrangian in the usual way

$$\begin{aligned}&{\rm Det}({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi)\\&\quad={\rm exp\,Tr}\ln({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi)\\&\quad=\exp{\rm i}\int{\rm d}^2x(-{\rm i}){\rm tr}\langle x|\ln({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi)|x\rangle\\&\quad=\exp{\rm i}\int{\rm d}^2x\tilde{\mathcal{L}}_{\rm eff}(x),\end{aligned} \qquad (3.6)$$

where

$$\tilde{\mathcal{L}}_{\rm eff}(x)=(-{\rm i}){\rm tr}\langle x|\ln({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi)|x\rangle. \qquad (3.7)$$

First, let us drop the contribution of gradients $\partial_{\mu}\sigma$ and $\partial_{\mu}\varphi$ and evaluate the effective potential $\tilde{V}[\sigma(x),\varphi(x)]=-\tilde{\mathcal{L}}_{\rm eff}(x)|_{\partial_{\mu}\sigma=\partial_{\mu}\varphi=0}$.

Dropping the contribution of the gradients $\partial_{\mu}\sigma$ and $\partial_{\mu}\varphi$, the evaluation of the functional determinant (3.5) runs in the following way

$$\begin{aligned}&{\rm Det}({\rm i}\gamma^{\mu}\partial_{\mu}-\sigma-{\rm i}\gamma^5\varphi)|_{\partial_{\mu}\sigma=\partial_{\mu}\varphi=0}\\&\quad={\rm Det}(\Box+\Phi^{\dagger}\Phi)\\&\quad=\exp{\rm i}\int{\rm d}^2x(-{\rm i}){\rm tr}\langle x|\ln(\Box+\Phi^{\dagger}\Phi)|x\rangle\\&\quad=\exp{\rm i}\int{\rm d}^2x\int\frac{{\rm d}^2k}{(2\pi)^2{\rm i}}\ln(-k^2+\Phi^{\dagger}(x)\Phi(x))\\&\quad=\exp{\rm i}\int{\rm d}^2x\int\frac{{\rm d}^2k_{\rm E}}{(2\pi)^2}\ln(k_{\rm E}^2+\Phi^{\dagger}(x)\Phi(x)),\end{aligned} \qquad (3.8)$$

where $\Phi(x)=\sigma(x)+{\rm i}\varphi(x)$ and $k_{\rm E}$ is the 2-momentum in Euclidean momentum space obtained from the 2-momentum k in Minkowski momentum space by means of a Wick rotation $k_0={\rm i}k_2$ [29]. The effective potential defined by the functional determinant (3.8) amounts to

$$\begin{aligned}
&-\tilde{V}[\sigma(x),\varphi(x)]\\
&\quad=\tilde{\mathcal{L}}_{\rm eff}(x)|_{\partial_\mu\sigma=\partial_\mu\varphi=0}\\
&\quad=\int\frac{{\rm d}^2k_{\rm E}}{(2\pi)^2}\ln(k^2_{\rm E}+\Phi^\dagger(x)\Phi(x))\\
&\quad=\frac{1}{4\pi}[(\Lambda^2+\Phi^\dagger(x)\Phi(x))\ln(\Lambda^2+\Phi^\dagger(x)\Phi(x))\\
&\qquad-\Phi^\dagger(x)\Phi(x)\ln\Phi^\dagger(x)\Phi(x)-\Lambda^2].
\end{aligned}\tag{3.9}$$

This result can be obtained differently by representing the effective Lagrangian (3.7) in terms of one-fermion loop diagrams [22]. Denoting $\tilde{\Phi}=\sigma+{\rm i}\gamma^5\varphi$ we get [22]

$$\begin{aligned}
\tilde{\mathcal{L}}_{\rm eff}(x)&=-{\rm itr}\left\langle x\left|\ln\left({\rm i}\gamma^\mu\partial_\mu-\tilde{\Phi}\right)\right|x\right\rangle\\
&=-{\rm itr}\langle x|\ln({\rm i}\gamma^\mu\partial_\mu)|x\rangle\\
&\quad+\sum^{\infty}_{n=1}\frac{\rm i}{n}{\rm tr}\left\langle x\left|\left(\frac{1}{{\rm i}\gamma^\mu\partial_\mu}\tilde{\Phi}\right)^n\right|x\right\rangle\\
&=-{\rm itr}\langle x|\ln({\rm i}\gamma^\mu\partial_\mu)|x\rangle+\sum^{\infty}_{n=1}\tilde{\mathcal{L}}^{(n)}_{\rm eff}(x),
\end{aligned}\tag{3.10}$$

where the effective Lagrangian $\tilde{\mathcal{L}}^{(n)}_{\rm eff}(x)$ is defined by [22]

$$\begin{aligned}
\tilde{\mathcal{L}}^{(n)}_{\rm eff}(x)&=\int\prod^{n-1}_{\ell}\frac{{\rm d}^2x_\ell{\rm d}^2k_\ell}{(2\pi)^2}{\rm e}^{-{\rm i}k_1\cdot x_1-{\rm i}k_2\cdot x_2-\ldots-{\rm i}k_n\cdot x}\\
&\quad\times\left(-\frac{1}{n}\frac{1}{4\pi}\right)\int\frac{{\rm d}^2k}{\pi{\rm i}}{\rm tr}\left\{\frac{1}{\hat{k}}\tilde{\Phi}(x_1)\frac{1}{\hat{k}+\hat{k}_1}\tilde{\Phi}(x_2)\right.\\
&\quad\left.\ldots\tilde{\Phi}(x_{n-1})\frac{1}{\hat{k}+\hat{k}_1+\ldots+\hat{k}_{n-1}}\tilde{\Phi}(x)\right\}
\end{aligned}\tag{3.11}$$

at $k_1+k_2+\ldots+k_n=0$.

Dropping the momenta $k_i(i=1,2,\ldots,n-1)$ giving the contributions of the gradients $\partial_\mu\sigma$ and $\partial_\mu\varphi$ in the effective Lagrangian we recast $\tilde{\mathcal{L}}^{(n)}_{\rm eff}(x)$ into the form

$$\begin{aligned}
\tilde{\mathcal{L}}^{(n)}_{\rm eff}(x)&=-\frac{1}{n}\frac{1}{4\pi}\\
&\quad\times\int\frac{{\rm d}^2k}{\pi{\rm i}}{\rm tr}\left\{\frac{1}{\hat{k}}\tilde{\Phi}(x)\frac{1}{\hat{k}}\tilde{\Phi}(x)\ldots\tilde{\Phi}(x)\frac{1}{\hat{k}}\tilde{\Phi}(x)\right\}.
\end{aligned}\tag{3.12}$$

A non-zero contribution comes only from even n, $n=2m(m=1,2,\ldots)$

$$\begin{aligned}
\tilde{\mathcal{L}}^{(2m)}_{\rm eff}(x)&=-\frac{1}{m}\frac{1}{4\pi}(\Phi^\dagger(x)\Phi(x))^m\int\frac{{\rm d}^2k}{\pi{\rm i}}\frac{1}{(k^2)^m}\\
&=\frac{(-1)^{m+1}}{m}\frac{1}{4\pi}(\Phi^\dagger(x)\Phi(x))^m\int^{\Lambda}_{\mu}\frac{{\rm d}k^2_{\rm E}}{(k^2_{\rm E})^m},
\end{aligned}\tag{3.13}$$

where Λ and μ are the ultra-violet and infra-red cut-offs. For $m=1$ we get

$$\tilde{\mathcal{L}}^{(2)}_{\rm eff}(x)=\Phi^\dagger(x)\Phi(x)\frac{1}{4\pi}\ln\frac{\Lambda^2}{\mu^2}.\tag{3.14}$$

In turn for $m\neq1$ we obtain

$$\begin{aligned}
\tilde{\mathcal{L}}^{(2m)}_{\rm eff}(x)&=\frac{(-1)^m}{m(m-1)}(\Phi^\dagger(x)\Phi(x))^m\\
&\quad\times\frac{1}{4\pi}\left[\left(\frac{1}{\Lambda^2}\right)^{m-1}-\left(\frac{1}{\mu^2}\right)^{m-1}\right].
\end{aligned}\tag{3.15}$$

The total effective Lagrangian is given by

$$\begin{aligned}
\tilde{\mathcal{L}}_{\rm eff}(x)&=-{\rm itr}\langle x|\ln({\rm i}\gamma^\mu\partial_\mu)|x\rangle+\Phi^\dagger(x)\Phi(x)\frac{1}{4\pi}\ln\frac{\Lambda^2}{\mu^2}\\
&\quad+\frac{1}{4\pi}\sum^{\infty}_{n=1}\frac{(-1)^{n+1}}{n(n+1)}(\Phi^\dagger(x)\Phi(x))^{n+1}\\
&\quad\times\left[\left(\frac{1}{\Lambda^2}\right)^n-\left(\frac{1}{\mu^2}\right)^n\right].
\end{aligned}\tag{3.16}$$

Summing up the infinite series we arrive at the expression

$$\begin{aligned}
\tilde{\mathcal{L}}_{\rm eff}(x)&=\frac{1}{4\pi}\left[(\Lambda^2\ln\Lambda^2-\Lambda^2-\mu^2\ln\mu^2+\mu^2)\right.\\
&\quad+\Phi^\dagger(x)\Phi(x)\ln\frac{\Lambda^2}{\mu^2}+(\Lambda^2+\Phi^\dagger(x)\Phi(x))\\
&\quad\times\ln\left(1+\frac{\Phi^\dagger(x)\Phi(x)}{\Lambda^2}\right)-(\mu^2+\Phi^\dagger(x)\Phi(x))\\
&\quad\left.\times\ln\left(1+\frac{\Phi^\dagger(x)\Phi(x)}{\mu^2}\right)\right],
\end{aligned}\tag{3.17}$$

where we have taken into account that (see (3.8))

$$\begin{aligned}
-{\rm itr}\langle x|\ln({\rm i}\gamma^\mu\partial_\mu)|x\rangle&=\frac{1}{4\pi}\int^{\Lambda}_{\mu}{\rm d}k^2_{\rm E}\ln k^2_{\rm E}\qquad(3.18)\\
&=\frac{1}{4\pi}(\Lambda^2\ln\Lambda^2-\Lambda^2-\mu^2\ln\mu^2+\mu^2).
\end{aligned}$$

Equation (3.17) can be simplified:

$$\begin{aligned}
\tilde{\mathcal{L}}_{\rm eff}(x)&=\frac{1}{4\pi}\left[(\mu^2-\Lambda^2)+(\Lambda^2+\Phi^\dagger(x)\Phi(x))\right.\\
&\quad\times\ln(\Lambda^2+\Phi^\dagger(x)\Phi(x))-(\mu^2+\Phi^\dagger(x)\Phi(x))\\
&\quad\left.\times\ln\left(\mu^2+\Phi^\dagger(x)\Phi(x)\right)\right].
\end{aligned}\tag{3.19}$$

Setting $\mu=0$ we arrive at the effective potential (3.9).

The total effective potential we obtain by summing up (3.9) and the quadratic term of (3.3) which has the form $(1/2g)\Phi^\dagger(x)\Phi(x)$:

$$\begin{aligned}
V[\Phi^\dagger(x)\Phi(x)]&=\tilde{V}[\Phi^\dagger(x)\Phi(x)]+\frac{1}{2g}\Phi^\dagger(x)\Phi(x)\\
&=\frac{1}{4\pi}\left[\Phi^\dagger(x)\Phi(x)\ln\Phi^\dagger(x)\Phi(x)\right.\\
&\quad-(\Lambda^2+\Phi^\dagger(x)\Phi(x))\ln(\Lambda^2+\Phi^\dagger(x)\Phi(x))\\
&\quad\left.+\frac{2\pi}{g}\Phi^\dagger(x)\Phi(x)+\Lambda^2\right].
\end{aligned}\tag{3.20}$$

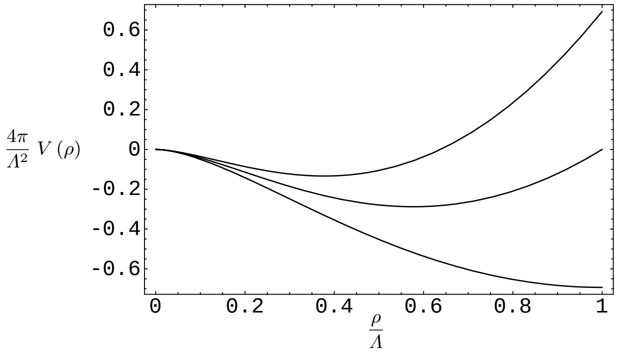

Fig. 1. The effective potential $V\,(\rho)$ of (3.21) as a function of ρ/Λ for $2\pi/g = \ln 2^k$ with $k = 1, 2$ and 3

In the polar representation $\Phi(x) = \rho(x)\mathrm{e}^{\mathrm{i}\vartheta(x)}$ corresponding to $\sigma(x) = \rho(x)\cos\vartheta(x)$ and $\varphi(x) = \rho(x)\sin\vartheta(x)$, the effective potential depends only on the ρ field and reads

$$V[\rho(x)] = \frac{1}{4\pi}\Big[\rho^2(x)\ln\frac{\rho^2(x)}{\Lambda^2} - (\Lambda^2+\rho^2(x))\ln\Big(1+\frac{\rho^2(x)}{\Lambda^2}\Big) + \frac{2\pi}{g}\rho^2(x)\Big], \quad (3.21)$$

where we have dropped the unimportant divergent contribution $(\Lambda^2 - \Lambda^2\ln\Lambda^2)/4\pi$. This shifts the effective potential to $V[0] = 0$.

It is well known that a quantum system has to be quantized around the minima of the effective potential. They are defined by[4]

$$\frac{\delta V[\bar{\rho}(x)]}{\delta\bar{\rho}(x)} = \frac{1}{2\pi}\bar{\rho}(x)\Big[-\ln\Big(1+\frac{\Lambda^2}{\bar{\rho}^2(x)}\Big) + \frac{2\pi}{g}\Big] = 0, \quad (3.22)$$

where $\bar{\rho}(x)$ is the vacuum expectation value of the ρ field, $\bar{\rho}(x) = \langle\rho(x)\rangle$. Equation (3.22) has a trivial solution $\bar{\rho}(x) = 0$ which corresponds to a maximum of the potential and a non-trivial one

$$\bar{\rho}(x) = \frac{\Lambda}{\sqrt{\mathrm{e}^{2\pi/g}-1}}. \quad (3.23)$$

The only constraint on the existence of the non-trivial solution is $g > 0$. This condition is trivial, since according to the analysis by Nambu and Jona–Lasinio [18] bound collective $\bar{\psi}\psi$ excitations can appear in a theory with the Lagrangian (1.18) only in the case of attraction between fermions, i.e. for positive g.

The effective potential $V\,(\rho)$ of (3.21) as a function of ρ/Λ is depicted in Fig. 1 for $2\pi/g = \ln 2^k$ with $k = 1, 2$ and 3. One can clearly see the maximum at $\bar{\rho} = 0$ and the minimum at $\bar{\rho}^2/\Lambda^2 = 1/(2^k - 1)$ corresponding to a non-trivial solution of the gap equation (3.22).

[4] The vacuum average $\langle 0|V[\rho(x)]|0\rangle$ of the effective potential we carry out in the tree approximation for the ρ field [19–22]. This yields $\langle 0|V[\rho(x)]|0\rangle_{\rm tree} = V[\langle\rho(x)\rangle]$

From the second derivative one can see that the effective potential (3.21) has a minimum only for the non-trivial solution of $\bar{\rho}(x)$ defined by (3.23). We denote this non-trivial solution $\bar{\rho}(x) = \rho_0$.

One can show that ρ_0 coincides with the dynamical mass M given by (1.15). To show this we derive the equations of motion

$$\bar{\psi}(x)\psi(x) = -\frac{\sigma(x)}{g},$$
$$\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x) = -\frac{\varphi(x)}{g}, \quad (3.24)$$

from the linearized Lagrangian defining the partition function $Z_{\rm Th}$ in (3.3).

The vacuum average $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle$ of the first equation of motion in the one-fermion loop approximation for the ψ field and in the tree approximation of the σ field gives

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\rm one\ loop} = -\frac{\langle 0|\sigma(x)|0\rangle_{\rm tree}}{g} = -\frac{\rho_0}{g}, \quad (3.25)$$

where

$$\langle 0|\sigma(x)|0\rangle_{\rm tree} = \langle\rho(x)\rangle\langle 0|\cos\vartheta(x)|0\rangle_{\rm tree} = \rho_0\cos\langle 0|\vartheta(x)|0\rangle = \rho_0.$$

Matching the r.h.s. of (3.25) with (1.16) for the fermion condensate one obtains $\rho_0 = M$. This demonstrates the complete agreement between the fermionic and bosonic description of the massless Thirring model. This result runs parallel to the dynamics of the evolution of the fermion system in NJL models describing well both low-energy interactions of hadrons [19–21] and confinement [22]. Below we would use M instead of ρ_0.

Expanding the effective potential around the minimum $\rho(x) = \rho_0 + \tilde{\rho}(x)$ we get

$$V[\tilde{\rho}(x)] = V[\rho_0] + \frac{1}{2\pi}(1-\mathrm{e}^{-2\pi/g})\tilde{\rho}^2(x) + \frac{1}{6\pi}\mathrm{e}^{-2\pi/g}(1-\mathrm{e}^{-2\pi/g})^{3/2}(1-2\mathrm{e}^{-2\pi/g})\frac{\tilde{\rho}^3(x)}{\Lambda} + O\Big(\frac{1}{\Lambda^2}\Big). \quad (3.26)$$

Keeping only terms surviving in the $\Lambda\to\infty$ limit we arrive at the expression

$$V[\tilde{\rho}(x)] = \frac{1}{2\pi}(1-\mathrm{e}^{-2\pi/g})\tilde{\rho}^2(x), \quad (3.27)$$

where we have dropped the trivial infinite constant $V[M]$.

It is clear from dimensional considerations that the gradient terms of the $\tilde{\rho}$ field $\partial_\mu\tilde{\rho}(x)$ appear in the effective Lagrangian only in the ratio $\partial_\mu\tilde{\rho}(x)/\Lambda$. Thereby, they vanish in the limit $\Lambda\to\infty$.

Hence, the effective potential defined by (3.27) implies that the fluctuations of the ρ field around the minimum (3.21) of the effective potential are described by a free

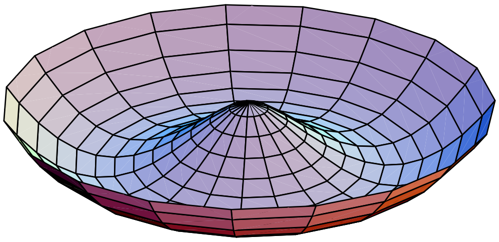

Fig. 2. The effective potential $V(\rho)$ of (3.21) as a function of ρ and ϑ for $2\pi/g = 2\ln 2$

scalar field $\tilde{\rho}(x)$ decoupled from the phase field $\vartheta(x)$[5]. The ρ and ϑ dependence of the effective potential $V(\rho)$ of (3.21) is shown in Fig. 2 for $2\pi/g = \ln 2^2$. For further investigation of the dynamics of the ϑ field one can integrate out the degrees of freedom related to the $\tilde{\rho}$ field and employ the representation

$$\Phi(x) = \sigma(x) + \mathrm{i}\varphi(x) = M\mathrm{e}^{\mathrm{i}\vartheta(x)}. \tag{3.28}$$

Hence, a bosonized version of the massless Thirring model obtained from the chirally broken phase of the fermion system is defined by only one degree of freedom, a scalar field $\vartheta(x)$.

In the approximation presented by (3.28) the partition function (3.4) reduces to the form

$$Z_{\mathrm{Th}} = \int \mathcal{D}\vartheta \mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu - M\mathrm{e}^{\mathrm{i}\gamma^5\vartheta}), \tag{3.29}$$

where we have dropped a trivial infinite constant. The functional determinant can be transformed as follows:

$$\begin{aligned}&\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu - M\mathrm{e}^{\mathrm{i}\gamma^5\vartheta})\\ &\quad = \mathrm{Det}(\mathrm{e}^{\mathrm{i}\gamma^5\vartheta/2}(\mathrm{i}\gamma^\mu\partial_\mu + \gamma^\mu A_\mu - M)\mathrm{e}^{\mathrm{i}\gamma^5\vartheta/2})\\ &\quad = J[\vartheta]\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu + \gamma^\mu A_\mu - M),\end{aligned} \tag{3.30}$$

where we have denoted

$$A_\mu(x) = \frac{1}{2}\varepsilon_{\mu\nu}\partial^\nu\vartheta(x). \tag{3.31}$$

The Jacobian $J[\vartheta]$ induced by a local chiral rotation can be calculated in the usual way [12–17]. In the appendix we show that by using an appropriate regularization scheme this Jacobian can be found to be equal to unity,

$$J[\vartheta] = 1. \tag{3.32}$$

The partition function (3.29) then reads

$$\begin{aligned}Z_{\mathrm{Th}} &= \int \mathcal{D}\vartheta \mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu + \gamma^\mu A_\mu - M)\\ &= \int \mathcal{D}\vartheta \exp \mathrm{i}\int \mathrm{d}^2x \mathcal{L}_{\mathrm{eff}}(x).\end{aligned} \tag{3.33}$$

The simplest way to calculate the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}(x)$ is to represent it in the form of one-fermion loop diagrams [19–22]. Since M is proportional to Λ, it is clear from dimensional considerations that the main contribution should come from the diagram with two vertices. The contribution of the diagram with $n > 2$ vertices falls as $O(1/\Lambda^{n-2})$ at $\Lambda \to \infty$. That is why the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}(x)$ is determined by

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}(x) &= -\mathrm{i}\langle x|\mathrm{tr}\ln(\mathrm{i}\gamma^\mu\partial_\mu - M)|x\rangle\\ &\quad - \frac{1}{8\pi}\int \frac{\mathrm{d}^2x_1\mathrm{d}^2k_1}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}k_1\cdot(x_1-x)}A_\mu(x)A_\nu(x_1)\\ &\quad \times \int\frac{\mathrm{d}^2k}{\pi\mathrm{i}}\mathrm{tr}\left\{\frac{1}{M-\hat{k}}\gamma^\mu\frac{1}{M-\hat{k}-\hat{k}_1}\gamma^\nu\right\}.\end{aligned} \tag{3.34}$$

Omitting a trivial infinite constant and keeping only the leading contribution at $\Lambda \to \infty$ we get

$$\mathcal{L}_{\mathrm{eff}}(x) = \frac{1}{16\pi}(1 - \mathrm{e}^{-2\pi/g})\partial_\mu\vartheta(x)\partial^\mu\vartheta(x), \tag{3.35}$$

where we have used the relation $\varepsilon_{\mu\alpha}\varepsilon^{\nu\alpha} = -g^\nu_\mu$.

This result testifies that the bosonized version of the massless Thirring model obtained from the chirally broken phase of the fermion system is a quantum field theory of a free massless scalar field $\vartheta(x)$.

4 Generating functional of Green functions in the massless Thirring model. Bosonization rules

Now we are able to turn to the problem of an explicit evaluation of arbitrary correlation functions in the massless Thirring model. To this aim we consider the generating functional of Green functions defined by

$$\begin{aligned}Z_{\mathrm{Th}}[J,\bar{J}] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\mathrm{i}\int\mathrm{d}^2x\Big\{\bar{\psi}(x)\mathrm{i}\gamma^\mu\partial_\mu\psi(x)\\ &\quad -\frac{1}{2}g\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(x)\gamma^\mu\psi(x)\\ &\quad +\bar{\psi}(x)J(x) + \bar{J}(x)\psi(x)\Big\}\\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\mathrm{i}\int\mathrm{d}^2x\Big\{\bar{\psi}(x)\mathrm{i}\gamma^\mu\partial_\mu\psi(x)\\ &\quad +\frac{1}{2}g[(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x))^2]\\ &\quad +\bar{\psi}(x)J(x) + \bar{J}(x)\psi(x)\Big\}\\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\mathcal{D}\varphi\exp\mathrm{i}\int\mathrm{d}^2x\Big\{\bar{\psi}(x)\mathrm{i}\gamma^\mu\partial_\mu\psi(x)\\ &\quad -\bar{\psi}(x)(\sigma(x) + \mathrm{i}\gamma^5\varphi(x))\psi(x) + \bar{\psi}(x)J(x)\\ &\quad +\bar{J}(x)\psi(x) - \frac{1}{2g}[\sigma^2(x) + \varphi^2(x)]\Big\},\end{aligned} \tag{4.1}$$

[5] The decoupling of the $\tilde{\rho}$ field is demonstrated in more detail in Appendix B

where $\bar{J}(x)$ and $J(x)$ are external sources of the Thirring fields $\psi(x)$ and $\bar{\psi}(x)$. Therewith, the external source $J(x)$, a column matrix with components $J_1(x)$ and $J_2(x)$, is responsible for the production of the $\psi^\dagger_2(x)$ and $\psi^\dagger_1(x)$ fields, whereas the external source $\bar{J}(x)$, a row matrix with components $J^\dagger_2(x)$ and $J^\dagger_1(x)$, produces the fields $\psi_1(x)$ and $\psi_2(x)$.

Integrating over the fermion fields we arrive at

$$\begin{aligned} Z_{\rm Th}[J,\bar{J}] &= \int \mathcal{D}\sigma\mathcal{D}\varphi {\rm Det}({\rm i}\gamma^\mu\partial_\mu - \sigma - {\rm i}\gamma^5\varphi) \\ &\times \exp {\rm i}\int {\rm d}^2x\left\{-\frac{1}{2g}[\sigma^2(x)+\varphi^2(x)]\right\} \\ &\times \exp {\rm i}\int {\rm d}^2x\left\{-\bar{J}(x)\frac{1}{{\rm i}\gamma^\mu\partial_\mu - \sigma(x) - {\rm i}\gamma^5\varphi(x)}J(x)\right\}. \end{aligned} \tag{4.2}$$

Skipping intermediate steps expounded in detail in Sect. 3 we get

$$\begin{aligned} Z_{\rm Th}[J,\bar{J}] &= \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x\bigg\{\frac{1}{16\pi}(1-{\rm e}^{-2\pi/g})\partial_\mu\vartheta(x) \\ &\times\partial^\mu\vartheta(x) - \bar{J}(x)\frac{1}{{\rm i}\gamma^\mu\partial_\mu - M{\rm e}^{{\rm i}\gamma^5\vartheta(x)}}J(x)\bigg\}. \end{aligned} \tag{4.3}$$

Keeping only leading terms in the $1/M$ expansion (or equivalently in $1/\Lambda$) we obtain

$$\begin{aligned} Z_{\rm Th}[J,\bar{J}] &= \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x\bigg\{\frac{1}{16\pi}(1-{\rm e}^{-2\pi/g})\partial_\mu\vartheta(x) \\ &\times\partial^\mu\vartheta(x) + \frac{1}{M}\bar{J}(x)\left(\frac{1-\gamma^5}{2}\right)J(x){\rm e}^{{\rm i}\vartheta(x)} \\ &+\frac{1}{M}\bar{J}(x)\left(\frac{1+\gamma^5}{2}\right)J(x){\rm e}^{-{\rm i}\vartheta(x)}\bigg\} \\ &= \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x\bigg\{\frac{1}{2}\frac{1}{8\pi}(1-{\rm e}^{-2\pi/g})\partial_\mu\vartheta(x) \\ &\times\partial^\mu\vartheta(x) + \frac{1}{M}J^\dagger_1(x)J_2(x){\rm e}^{{\rm i}\vartheta(x)} \\ &+\frac{1}{M}J^\dagger_2(x)J_1(x){\rm e}^{-{\rm i}\vartheta(x)}\bigg\}. \end{aligned} \tag{4.4}$$

By normalizing the ϑ field, $\vartheta(x)\to\beta\vartheta(x)$, with β given by the condition

$$\frac{8\pi}{\beta^2} = 1 - {\rm e}^{-2\pi/g}, \tag{4.5}$$

resembling Coleman's relation [3], and defining correctly the kinetic term of the renormalized field $\vartheta(x)$, we arrive at

$$\begin{aligned} Z_{\rm Th}[J,\bar{J}] &= \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x\left\{\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{1}{M}\right. \\ &\left.\times J^\dagger_1(x)J_2(x){\rm e}^{{\rm i}\beta\vartheta(x)} + \frac{1}{M}J^\dagger_2(x)J_1(x){\rm e}^{-{\rm i}\beta\vartheta(x)}\right\}. \end{aligned} \tag{4.6}$$

The vacuum expectation value of the fermion fields considered by Coleman [3],

$$\begin{aligned} &\left\langle 0\left|T\left(\prod^n_{k=1}\sigma_+(x_k)\sigma_-(y_k)\right)\right|0\right\rangle \\ &= \left\langle 0\left|T\left(\prod^n_{k=1}[\psi^\dagger_2(x_k)\psi_1(x_k)][\psi^\dagger_1(y_k)\psi_2(y_k)]\right)\right|0\right\rangle, \end{aligned} \tag{4.7}$$

can be represented in the form of functional derivatives with respect to external sources $J^\dagger_1(x)$, $J^\dagger_2(x)$ and $J_1(x)$, $J_2(x)$:

$$\begin{aligned} &\left\langle 0\left|T\left(\prod^n_{k=1}\sigma_+(x_k)\sigma_-(y_k)\right)\right|0\right\rangle \\ &= \left\langle 0\left|T\left(\prod^n_{k=1}[\psi^\dagger_2(x_k)\psi_1(x_k)][\psi^\dagger_1(y_k)\psi_2(y_k)]\right)\right|0\right\rangle \\ &= \prod^n_{k=1}\frac{\delta}{{\rm i}\delta J_1(x_k)}\frac{\delta}{{\rm i}\delta J^\dagger_2(x_k)}\frac{\delta}{{\rm i}\delta J_2(y_k)}\frac{\delta}{{\rm i}\delta J^\dagger_1(y_k)} \\ &\times Z_{\rm Th}[J,\bar{J}]|_{J_1=J_2=J^\dagger_1=J^\dagger_2=0}. \end{aligned} \tag{4.8}$$

Using the generating functional $Z_{\rm Th}[J,\bar{J}]$ in the form (4.6) the r.h.s. of (4.8) can be written as

$$\begin{aligned} &\left\langle 0\left|T\left(\prod^n_{k=1}\sigma_+(x_k)\sigma_-(y_k)\right)\right|0\right\rangle \\ &= \left\langle 0\left|T\left(\prod^n_{k=1}[\psi^\dagger_2(x_k)\psi_1(x_k)][\psi^\dagger_1(y_k)\psi_2(y_k)]\right)\right|0\right\rangle \\ &= (-1)^n\left(\frac{\delta^2(0)}{M}\right)^{2n}\left\langle 0\left|T\left(\prod^n_{i=1}[A_-(x_i)A_+(y_i)]\right)\right|0\right\rangle, \end{aligned} \tag{4.9}$$

where $\delta^2(0) = \int {\rm d}^2p/(2\pi)^2 = {\rm i}\int {\rm d}^2p_{\rm E}/(2\pi)^2 = {\rm i}\bar{\Lambda}^2/4\pi$. The cut-off $\bar{\Lambda}$ is invented to regularize divergences coming from the closed loops of the ϑ field. The two-point Green function of the ϑ field (2.9) regularized at $x=y$ is defined by

$${\rm i}\Delta(0) = -\frac{1}{4\pi}\ln\left(\frac{\bar{\Lambda}^2}{\mu^2}\right). \tag{4.10}$$

Thereby, the relation (4.9) can be rewritten as follows:

$$\begin{aligned} &\left\langle 0\left|T\left(\prod^n_{k=1}\sigma_+(x_k)\sigma_-(y_k)\right)\right|0\right\rangle \\ &= \left\langle 0\left|T\left(\prod^n_{k=1}[\psi^\dagger_2(x_k)\psi_1(x_k)][\psi^\dagger_1(y_k)\psi_2(y_k)]\right)\right|0\right\rangle \\ &= \left(\frac{\bar{\Lambda}^2}{4\pi M}\right)^{2n}\left\langle 0\left|T\left(\prod^n_{i=1}[A_-(x_i)A_+(y_i)]\right)\right|0\right\rangle, \end{aligned} \tag{4.11}$$

Relation (4.11) demonstrates the equivalence between vacuum expectation values in the massless Thirring model and vacuum expectation values in a quantum field theory of a massless scalar field $\vartheta(x)$ coupled to external sources via exponential couplings $A_+(x) = \mathrm{e}^{\mathrm{i}\beta\vartheta(x)}$ and $A_-(x) = \mathrm{e}^{-\mathrm{i}\beta\vartheta(x)}$ (see (2.3)).

In order to fix the value of $\bar{\Lambda}$ in terms of M we suggest to evaluate the vacuum expectation value of the operator

$$(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x))^2 = 4\sigma_+(x)\sigma_-(x). \quad (4.12)$$

In Sect. 6 (see (6.54)) we show that this operator is an integral of motion and is equal to M^2/g^2. The evaluation of the vacuum expectation value of the operator (4.12) can be carried out with the help of (4.11). The result reads

$$\begin{aligned}
&\langle 0\,|[(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x))^2]\,|\,0\rangle \\
&\quad = \langle 0\,|[4\sigma_+(x)\sigma_-(x)]|\,0\rangle \\
&\quad = \left(\frac{\bar{\Lambda}^2}{2\pi M}\right)^2 \langle 0\,|[A_-(x)A_+(x)]|\,0\rangle \\
&\quad = \left(\frac{\bar{\Lambda}^2}{2\pi M}\right)^2. \qquad (4.13)
\end{aligned}$$

Equating the r.h.s. of (4.13) to M^2/g^2 we obtain the cut-off $\bar{\Lambda}$ in terms of M and g

$$\bar{\Lambda} = \sqrt{\frac{2\pi}{g}}M. \quad (4.14)$$

In the strong coupling limit $g \to \infty$ we get $\bar{\Lambda} \to \Lambda$, whereas in the weak coupling limit $g \to 0$ the cut-off $\bar{\Lambda}$ vanishes. The former corresponds to the absence of the ϑ field fluctuations in a free massless fermion field theory.

Using the relation (4.14) we recast the r.h.s. of (4.11) into the form

$$\begin{aligned}
&\left\langle 0\left|T\left(\prod_{k=1}^{n}\sigma_+(x_k)\sigma_-(y_k)\right)\right|0\right\rangle \\
&\quad = \left\langle 0\left|T\left(\prod_{k=1}^{n}[\psi_2^\dagger(x_k)\psi_1(x_k)][\psi_1^\dagger(y_k)\psi_2(y_k)]\right)\right|0\right\rangle \\
&\quad = \frac{\langle\bar{\psi}\psi\rangle^{2n}}{2^{2n}}\left\langle 0\left|T\left(\prod_{i=1}^{n}[A_-(x_i)A_+(y_i)]\right)\right|0\right\rangle, \quad (4.15)
\end{aligned}$$

where we have used that $\langle\bar{\psi}\psi\rangle = -M/g$ (1.16).

Using (2.10) the r.h.s. of (4.15) can be calculated explicitly and reads

$$\begin{aligned}
&\left\langle 0\left|T\left(\prod_{k=1}^{n}\sigma_+(x_k)\sigma_-(y_k)\right)\right|0\right\rangle \\
&\quad = \frac{\langle\bar{\psi}\psi\rangle^{2n}}{2^{2n}}\mathrm{e}^{n\beta^2\mathrm{i}\Delta(0)} \qquad (4.16) \\
&\quad \times\frac{\prod\limits_{j<k}^{n}[-\mu^2(x_j-x_k)^2]^{\beta^2/4\pi}[-\mu^2(y_j-y_k)^2]^{\beta^2/4\pi}}{\prod_{j=1}^{n}\prod_{k=1}^{n}[-\mu^2(x_j-y_k)^2]^{\beta^2/4\pi}},
\end{aligned}$$

where $\mathrm{i}\Delta(0)$ is defined by (4.10). Formula (4.16) reproduces, in principle, Klaiber's equations [5] used further by Coleman [3] but with a relation between the coupling constants β and g (4.5) different to that suggested by Coleman (1.9) [3]. The new relation (4.5) is caused by the fact that in our approach unlike in that of Coleman the fermion system is in the chirally broken phase.

Relation (4.15) between the vacuum expectation values can be represented in operator form by the Abelian bosonization rules

$$Z\bar{\psi}(x)\left(\frac{1\mp\gamma^5}{2}\right)\psi(x) = \frac{1}{2}\langle\bar{\psi}\psi\rangle\mathrm{e}^{\pm\mathrm{i}\beta\vartheta(x)}. \quad (4.17)$$

They can be derived straightforwardly from the equations of motion (3.24) for $\sigma(x)$ and $\varphi(x)$ connected by (3.28), where $M/g = -\langle\bar{\psi}\psi\rangle$,

$$\bar{\psi}(x)\left(\frac{1\mp\gamma^5}{2}\right)\psi(x) = \frac{1}{2}\langle\bar{\psi}\psi\rangle\mathrm{e}^{\pm\mathrm{i}\beta\vartheta(x)}, \quad (4.18)$$

with a subsequent renormalization of the fermion field $\psi(x) \to Z^{1/2}\psi(x)$, where Z is a renormalization constant. The parameter Z is invented to remove divergences appearing in the evaluation of the vacuum expectation values of $A_-(x)$ and $A_+(y)$. If such divergences do not appear the parameter Z should be set unity, $Z = 1$. For example, in one-loop approximation for the fermion field and tree approximation for the ϑ field one obtains $Z = 1$.

Relation (4.17) is analogous to the Abelian bosonization rules derived by Coleman (1.10) in the massive Thirring model. For the massless Thirring model due to the employment of the chiral symmetric phase with a chiral symmetric vacuum giving $\langle\bar{\psi}\psi\rangle = 0$ Coleman's procedure fails in deriving a relation like (4.17).

In Sect. 6 we show that the Abelian bosonization rules (4.17) are consistent with the equations of motion for fermionic fields evolving out of the chirally broken phase.

Using the Abelian bosonization rules (4.17) we are able to evaluate the vacuum expectation value of the $\bar{\psi}(x)\psi(x)$ operator:

$$\begin{aligned}
\langle 0|\bar{\psi}(x)\psi(x)|0\rangle &= \langle 0|[\sigma_+(x)+\sigma_-(x)]|0\rangle \\
&= \frac{1}{2}Z^{-1}\langle\bar{\psi}\psi\rangle\langle 0|[A_-(x)+A_+(x)]|0\rangle \\
&= Z^{-1}\langle\bar{\psi}\psi\rangle\langle 0|\cos\beta\vartheta(x)|0\rangle = 0, \quad (4.19)
\end{aligned}$$

where we have used (2.10) and (2.11).

We would like to emphasize that the vanishing of the vacuum expectation value (4.19) is caused by the infrared behavior of the ϑ field. This is related to the $\mu \to 0$ limit which takes into account long-range fluctuations. In this region the ϑ field is ill-defined [26,27] which leads to the randomization of the ϑ field in the infrared region [28]. Due to this $\cos\beta\vartheta(x)$ is averaged to zero [28]. This result agrees with the Mermin–Wagner theorem [25] pointing out the absence of long-range order in two-dimensional models. However, since the randomization of the ϑ field in the infrared region is fully a 1 + 1-dimensional problem, one can avoid the vanishing of $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle$ by means of dimensional regularization. In more detail we discuss this problem in Sect. 8. There we give also an exact solution for the massless Thirring model in the sense of the evaluation of any correlation function.

5 Bosonization of the massive Thirring model

The massive Thirring model differs from the massless model by the term $-m\bar{\psi}(x)\psi(x)$ in the Lagrangian, where m is the fermion mass.

Skipping intermediate steps which we have carried out explicitly in Sect. 2 we arrive at the partition function of the massive Thirring model given in terms of the path integral over fermion fields and over fields of collective excitations

$$\begin{aligned} Z_{\rm Th} = &\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\mathcal{D}\varphi \\ &\times \exp{\rm i}\int {\rm d}^2x\Big\{\bar{\psi}(x){\rm i}\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x) \\ &- \bar{\psi}(x)(\sigma(x) + {\rm i}\gamma^5\varphi(x))\psi(x) \\ &- \frac{1}{2g}[\sigma^2(x) + \varphi^2(x)]\Big\}. \end{aligned} \quad (5.1)$$

By a shift of the σ field, $m + \sigma \to \sigma$, we obtain

$$\begin{aligned} Z_{\rm Th} = &\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\mathcal{D}\varphi\exp{\rm i}\int {\rm d}^2x\Big\{\bar{\psi}(x){\rm i}\gamma^\mu\partial_\mu\psi(x) \\ &- \bar{\psi}(x)(\sigma(x) + {\rm i}\gamma^5\varphi(x))\psi(x) \\ &- \frac{1}{2g}[\sigma^2(x) + \varphi^2(x)] + \frac{m}{g}\sigma(x)\Big\}, \end{aligned} \quad (5.2)$$

where we have dropped an infinite constant proportional to m^2.

Integrating over fermionic degrees of freedom we recast the integrand into the form

$$\begin{aligned} Z_{\rm Th} = &\int \mathcal{D}\sigma\mathcal{D}\varphi{\rm Det}({\rm i}\gamma^\mu\partial_\mu - \sigma - {\rm i}\gamma^5\varphi) \quad (5.3) \\ &\times \exp{\rm i}\int {\rm d}^2x\left\{-\frac{1}{2g}[\sigma^2(x) + \varphi^2(x)] + \frac{m}{g}\sigma(x)\right\}. \end{aligned}$$

Since the functional determinant coincides completely with the determinant calculated in Sect. 3, we can immediately write down the total effective potential

$$\begin{aligned} &V[\Phi^\dagger(x), \Phi(x)] \\ &= \tilde{V}[\Phi^\dagger(x)\Phi(x)] + \frac{1}{2g}\Phi^\dagger(x)\Phi(x) - \frac{m}{g}\sigma(x) \\ &= \frac{1}{4\pi}\Big[\Phi^\dagger(x)\Phi(x)\ln\Phi^\dagger(x)\Phi(x) \\ &-(\Lambda^2 + \Phi^\dagger(x)\Phi(x))\ln(\Lambda^2 + \Phi^\dagger(x)\Phi(x)) \\ &+\frac{2\pi}{g}\Phi^\dagger(x)\Phi(x) - \frac{4\pi m}{g}\sigma(x) + \Lambda^2\Big]. \end{aligned} \quad (5.4)$$

In polar representation the effective potential (5.4) up to an infinite constant takes the form

$$\begin{aligned} V[\rho(x), \vartheta(x)] = &\frac{1}{4\pi}\Big[\rho^2(x)\ln\frac{\rho^2(x)}{\Lambda^2} - (\Lambda^2 + \rho^2(x)) \\ &\times\ln\left(1 + \frac{\rho^2(x)}{\Lambda^2}\right) + \frac{2\pi}{g}\rho^2(x) - \frac{4\pi m}{g}\rho(x) \\ &-\frac{4\pi m}{g}\rho(x)(\cos\vartheta(x) - 1)\Big]. \end{aligned} \quad (5.5)$$

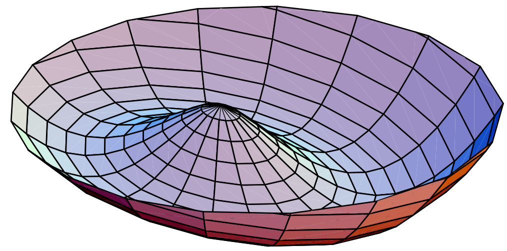

Fig. 3. The effective potential $V(\rho)$ of (5.5) as a function of ρ and ϑ for $2\pi/g = \ln 2^2$ and $4\pi m/g = 0.2$ in units of Λ

A graphical representation of this potential as a function of ρ and ϑ is shown in Fig. 3 for $2\pi/g = \ln 2^2$ and $4\pi m/g = 0.2$ in units of Λ.

For the calculation of the minimum of the effective potential (5.5) we have to calculate the vacuum expectation value

$$\begin{aligned} V[\bar{\rho}(x), 0] = &\frac{1}{4\pi}\Big[\bar{\rho}^2(x)\ln\frac{\bar{\rho}^2(x)}{\Lambda^2} - (\Lambda^2 + \bar{\rho}^2(x)) \quad (5.6) \\ &\times\ln\left(1 + \frac{\bar{\rho}^2(x)}{\Lambda^2}\right) + \frac{2\pi}{g}\bar{\rho}^2(x) - \frac{4\pi m}{g}\bar{\rho}(x)\Big], \end{aligned}$$

where we have used $\langle 0|\vartheta(x)|0\rangle = 0$ and $\langle 0|\cos\vartheta(x) - 1|0\rangle = \cos\langle 0|\vartheta(x)|0\rangle - 1 = 0$ which corresponds to the tree approximation for the ϑ field. The first derivative of the effective potential (5.6) with respect to $\bar{\rho}(x)$ is given by

$$\begin{aligned} &\frac{\delta V[\bar{\rho}(x), 0]}{\delta\bar{\rho}(x)} \\ &= \frac{1}{\pi}\bar{\rho}(x)\left[-\ln\left(1 + \frac{\Lambda^2}{\bar{\rho}^2(x)}\right) + \frac{2\pi}{g} - \frac{2\pi m}{g}\frac{1}{\bar{\rho}(x)}\right] \\ &= 0. \end{aligned} \quad (5.7)$$

The r.h.s of (5.7) can be rewritten in a more convenient form:

$$\bar{\rho}(x) = m + \bar{\rho}(x)\frac{g}{2\pi}\ln\left(1 + \frac{\Lambda^2}{\bar{\rho}^2(x)}\right). \quad (5.8)$$

This result agrees well with the gap equation (1.14) modified for $m \neq 0$,

$$M = m + M\frac{g}{2\pi}\ln\left(1 + \frac{\Lambda^2}{M^2}\right), \quad (5.9)$$

with $\bar{\rho}(x) = M$. By using (1.16) relation (5.9) reads

$$M - m = -g\langle\bar{\psi}\psi\rangle. \quad (5.10)$$

The solution of (5.8) is equal to

$$\begin{aligned} \bar{\rho}(x) &= M \\ &= \frac{\Lambda}{\sqrt{{\rm e}^{2\pi/g} - 1}} + \frac{\pi}{g}\frac{m}{1 - {\rm e}^{-2\pi/g}} + O\left(\frac{m^2}{\Lambda}\right). \end{aligned} \quad (5.11)$$

Since $\Lambda \gg m$, our statement concerning the decoupling of the $\tilde{\rho}$ field is also valid for the bosonization of the massive Thirring model. This implies that the bosonized version of the massive Thirring model as well as the massless one is described by one degree of freedom, the scalar field $\vartheta(x)$.

The partition function of the bosonized version of the massive Thirring model defined in the vicinity of the minimum of the effective potential (5.5) acquires the form

$$\begin{aligned} Z_{\rm Th} &= \int \mathcal{D}\vartheta {\rm Det}({\rm i}\gamma^\mu\partial_\mu - M{\rm e}^{{\rm i}\gamma^5\vartheta}) \\ &\times \exp {\rm i}\int {\rm d}^2x \frac{mM}{g}(\cos\vartheta(x) - 1) \\ &= \int \mathcal{D}\vartheta {\rm Det}({\rm i}\gamma^\mu\partial_\mu + \gamma^\mu A_\mu - M) \\ &\times \exp {\rm i}\int {\rm d}^2x \frac{mM}{g}(\cos\vartheta(x) - 1) \\ &= \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x \mathcal{L}_{\rm eff}(x), \end{aligned} \quad (5.12)$$

where $A_\mu(x) = (1/2)\varepsilon_{\mu\nu}\partial^\nu\vartheta(x)$ and the effective Lagrangian $\mathcal{L}_{\rm eff}(x)$ is determined by

$$\begin{aligned} \mathcal{L}_{\rm eff}(x) &= -{\rm i}\langle x|{\rm tr}\ln({\rm i}\gamma^\mu\partial_\mu - M)|x\rangle \\ &- \frac{1}{8\pi}\int \frac{{\rm d}^2x_1{\rm d}^2k_1}{(2\pi)^2}{\rm e}^{-{\rm i}k_1\cdot(x_1-x)}A_\mu(x)A_\nu(x_1) \\ &\times \int \frac{{\rm d}^2k}{\pi {\rm i}}{\rm tr}\left\{\frac{1}{M-\hat{k}}\gamma^\mu\frac{1}{M-\hat{k}-\hat{k}_1}\gamma^\nu\right\} \\ &+ \frac{mM}{g}(\cos\vartheta(x) - 1) \end{aligned} \quad (5.13)$$

in complete analogy with the massless case (3.34).

Omitting a trivial infinite constant and the terms proportional to inverse powers of Λ leads to

$$\begin{aligned} \mathcal{L}_{\rm eff}(x) &= \frac{1}{16\pi}(1 - {\rm e}^{-2\pi/g})\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) \\ &+ \frac{mM}{g}(\cos\vartheta(x) - 1). \end{aligned} \quad (5.14)$$

In order to get the correct kinetic term of the ϑ field, we renormalize the ϑ field, $\vartheta(x) \to \beta\vartheta(x)$, where the renormalization constant β obeys relation (4.5). Introducing a parameter α

$$\alpha = \frac{mM}{g} = -m\beta^2\langle\bar{\psi}\psi\rangle + \frac{m^2}{g}\beta^2, \quad (5.15)$$

where we have used (5.10), we transform the effective Lagrangian (5.14) to the standard form of the Lagrangian of the SG model [3]

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1). \quad (5.16)$$

This testifies the complete equivalence of the bosonized version of the massive Thirring model and the SG model

$$Z_{\rm Th} = Z_{\rm SG}, \quad (5.17)$$

with the relation (4.5) between the coupling constants β and g.

The generating functional of the Green functions $Z_{\rm Th}[J,\bar{J}]$ in the massive Thirring model can be derived in analogy to (4.6), the generating functional of the Green functions in the massless Thirring model, and reads

$$\begin{aligned} Z_{\rm Th}[J,\bar{J}] = \int \mathcal{D}\vartheta \exp {\rm i}\int {\rm d}^2x\Big\{&\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) \\ &+\frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1) + \frac{1}{M}J^\dagger_1(x)J_2(x){\rm e}^{{\rm i}\beta\vartheta(x)} \\ &+\frac{1}{M}J^\dagger_2(x)J_1(x){\rm e}^{-{\rm i}\beta\vartheta(x)}\Big\}. \end{aligned} \quad (5.18)$$

The Abelian bosonization rules analogous to (4.17) in the massless Thirring model can be derived from the equations of motion

$$\begin{aligned} \bar{\psi}(x)\psi(x) &= -\frac{\sigma(x) - m}{g}, \\ \bar{\psi}(x){\rm i}\gamma^5\psi(x) &= -\frac{\varphi(x)}{g}. \end{aligned} \quad (5.19)$$

Setting $\sigma(x) = M\cos\beta\vartheta(x)$ and $\varphi(x) = M\sin\beta\vartheta(x)$ we get

$$m\bar{\psi}(x)\left(\frac{1\mp\gamma^5}{2}\right)\psi(x) = -\frac{\alpha}{2\beta^2}{\rm e}^{\pm{\rm i}\beta\vartheta(x)} + \frac{m^2}{2g}. \quad (5.20)$$

Renormalizing the fermion field $\psi(x) \to Z^{1/2}\psi(x)$ we arrive at the relation

$$Zm\bar{\psi}(x)\left(\frac{1\mp\gamma^5}{2}\right)\psi(x) = -\frac{\alpha}{2\beta^2}{\rm e}^{\pm{\rm i}\beta\vartheta(x)} + \frac{m^2}{2g}. \quad (5.21)$$

For the evaluation of the vacuum expectation value in one-loop approximation for the fermion field and in tree approximation for the scalar field, the parameter Z amounts to $Z = 1$.

The operator relation (5.21) can be considered as a generalization of the Abelian bosonization rules (1.10) derived by Coleman. The term proportional to m^2 can be dropped at leading order in the m expansion [3].

We would like to accentuate that in our case the coupling constant β^2 is always greater than 8π, $\beta^2 > 8\pi$. This is in disagreement with Coleman's statement pointing out that the equivalence between the massive Thirring model and the SG model can exist only if $\beta^2 < 8\pi$ [3]. Such a disagreement can be explained by different starting phases of the fermion system evolving to the bosonic phase. In fact, in Coleman's approach the fermion system has been considered in the chiral symmetric phase, whereas in our case the fermion system is in the phase of spontaneously broken chiral symmetry. We would like to recall that in the Thirring model with an attractive four-fermion interaction the chirally broken phase is preferable.

6 Operator formalism for the massless Thirring model

In this section we investigate the massless Thirring model in the operator formalism. We analyze the normal ordering of fermion operators and chiral symmetry breaking, the equations of motion for fermion fields, the current algebra and the connection of the energy-momentum tensor $\theta_{\mu\nu}(x,t)$ [30] with its Sugawara form [31,32]. We show that the Schwinger term [33] calculated for the fermion system in the chirally broken phase depends on the coupling constant g and reduces to the result obtained by Sommerfield [34] in the limit $g \to 0$. We demonstrate that Sommerfield's value for the Schwinger term corresponds to a trivial vacuum of the fermion system.

We discuss the phenomenon of spontaneous breaking of chiral symmetry in the massless Thirring case from the point of view of the BCS theory of superconductivity. We use the BCS expression for the wave function of the non-perturbative vacuum and calculate the energy density of this non-perturbative vacuum state. We show that the energy density of the non-perturbative vacuum acquires a minimum just, when the dynamical mass M of the fermions satisfies the gap equation (1.14).

6.1 Normal ordering and chiral symmetry breaking

The Lagrangian of the massless Thirring that we use below reads

$$\mathcal{L}(x,t) = :\bar{\psi}(x,t)\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi(x,t): - \frac{1}{2}g :\bar{\psi}(x,t)\gamma_{\mu}\psi(x,t)\bar{\psi}(x,t)\gamma^{\mu}\psi(x,t):, \quad (6.1)$$

where $:\ldots:$ denotes normal ordering. A vector current $j_{\mu}(x,t)$ and the divergence $\partial^{\mu}j_{\mu}(x,t)$ can be derived in the usual way by a local gauge transformation $U_{\mathrm{V}}(1)$ with a parameter $\alpha_{\mathrm{V}}(x,t)$:

$$\begin{aligned}\psi(x,t) &\to \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{V}}(x,t)}\psi(x,t),\\ \bar{\psi}(x,t) &\to \bar{\psi}(x,t)\mathrm{e}^{-\mathrm{i}\alpha_{\mathrm{V}}(x,t)}.\end{aligned} \quad (6.2)$$

This changes the Lagrangian (6.1) as follows:

$$\mathcal{L}(x,t) \to \mathcal{L}[\alpha_{\mathrm{V}}(x,t)] = \mathcal{L}(x,t) - :\bar{\psi}(x,t)\gamma_{\mu}\psi(x,t): \partial^{\mu}\alpha_{\mathrm{V}}(x,t). \quad (6.3)$$

Therefore, the vector current $j_{\mu}(x,t)$ and its divergence $\partial^{\mu}j_{\mu}(x,t)$ are equal to

$$\begin{aligned}j_{\mu}(x,t) &= -\frac{\delta\mathcal{L}[\alpha_{\mathrm{V}}(x,t)]}{\delta\partial^{\mu}\alpha_{\mathrm{V}}(x,t)} =: \bar{\psi}(x,t)\gamma_{\mu}\psi(x,t):,\\ \partial^{\mu}j_{\mu}(x,t) &= -\frac{\delta\mathcal{L}[\alpha_{\mathrm{V}}(x,t)]}{\delta\alpha_{\mathrm{V}}(x,t)} = 0.\end{aligned} \quad (6.4)$$

For the subsequent analysis we need the interaction term in the Lagrangian (6.1) in the form of a product of currents $j_{\mu}(x,t)j^{\mu}(x,t)$. In order to understand the replacement

$$:\bar{\psi}(x,t)\gamma_{\mu}\psi(x,t)\bar{\psi}(x,t)\gamma^{\mu}\psi(x,t): \to j_{\mu}(x,t)j^{\mu}(x,t)$$

we suggest to start with the product $j_{\mu}(x,t)j^{\mu}(x,t)$ and Wick's theorem to reduce this product to the form of the interaction term in (6.1). It is useful to employ Schwinger's method of separation [33]. Denoting $(x,t) \to x$ we obtain

$$\begin{aligned}j_{\mu}(x)j^{\mu}(x) &= :\bar{\psi}(x)\gamma_{\mu}\psi(x)::\bar{\psi}(x)\gamma^{\mu}\psi(x): \\ &= \lim_{\varepsilon\to 0} :\bar{\psi}\left(x+\frac{1}{2}\varepsilon\right)\gamma_{\mu}\psi\left(x+\frac{1}{2}\varepsilon\right): \\ &\quad\times :\bar{\psi}\left(x-\frac{1}{2}\varepsilon\right)\gamma^{\mu}\psi\left(x-\frac{1}{2}\varepsilon\right): \\ &= \lim_{\varepsilon\to 0}\Bigg[:\bar{\psi}\left(x+\frac{1}{2}\varepsilon\right)\gamma_{\mu}\psi\left(x+\frac{1}{2}\varepsilon\right) \\ &\quad\times\bar{\psi}\left(x-\frac{1}{2}\varepsilon\right)\gamma^{\mu}\psi\left(x-\frac{1}{2}\varepsilon\right): \\ &\quad + :\bar{\psi}\left(x+\frac{1}{2}\varepsilon\right)\gamma_{\mu}\left\langle 0\left|\psi\left(x+\frac{1}{2}\varepsilon\right)\right.\right. \\ &\quad\times\left.\left.\bar{\psi}\left(x-\frac{1}{2}\varepsilon\right)\right|0\right\rangle\gamma^{\mu}\psi\left(x-\frac{1}{2}\varepsilon\right): \\ &\quad + :\bar{\psi}\left(x-\frac{1}{2}\varepsilon\right)\gamma_{\mu}\left\langle 0\left|\psi\left(x-\frac{1}{2}\varepsilon\right)\right.\right. \\ &\quad\times\left.\left.\bar{\psi}\left(x+\frac{1}{2}\varepsilon\right)\right|0\right\rangle\gamma^{\mu}\psi\left(x+\frac{1}{2}\varepsilon\right): \\ &\quad -\mathrm{tr}\left\{\gamma_{\mu}\left\langle 0\left|\psi\left(x-\frac{1}{2}\varepsilon\right)\bar{\psi}\left(x+\frac{1}{2}\varepsilon\right)\right|0\right\rangle\right. \\ &\quad\times\left.\gamma^{\mu}\left\langle 0\left|\psi\left(x+\frac{1}{2}\varepsilon\right)\bar{\psi}\left(x-\frac{1}{2}\varepsilon\right)\right|0\right\rangle\right\}\Bigg],\end{aligned} \quad (6.5)$$

where $\varepsilon = (\varepsilon^0, \varepsilon^1)$ is an infinitesimal 2-vector.

Now we would like to discuss the contributions caused by the vacuum expectation values in (6.5). For the free massless fermion field we get

$$\begin{aligned}&\left\langle 0\left|\psi\left(x\pm\frac{1}{2}\varepsilon\right)\bar{\psi}\left(x\mp\frac{1}{2}\varepsilon\right)\right|0\right\rangle \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{4\pi}\frac{\gamma^0|p|-\gamma^1 p}{|p|}\mathrm{e}^{\mp\mathrm{i}(|p|\varepsilon^0 - p\varepsilon^1)} \\ &= \hat{\varepsilon}\left[\delta(\varepsilon^2) \mp \frac{\mathrm{i}}{2\pi}\frac{1}{\varepsilon^2}\right].\end{aligned} \quad (6.6)$$

Due to the identity $\gamma_{\mu}\gamma^{\alpha}\gamma^{\mu} = 0$ for $\alpha = 0,1$ these vacuum expectation values taken between γ matrices $\gamma_{\mu}\ldots\gamma^{\mu}$ vanish:

$$\gamma_{\mu}\left\langle 0\left|\psi\left(x\pm\frac{1}{2}\varepsilon\right)\bar{\psi}\left(x\mp\frac{1}{2}\varepsilon,\right)\right|0\right\rangle\gamma^{\mu} = 0. \quad (6.7)$$

This result persists for the interacting massless fermion field. In fact, the vacuum expectation values calculated for the trivial vacuum should have the following general form:

$$\left\langle 0\left|\psi\left(x\pm\frac{1}{2}\varepsilon\right)\bar{\psi}\left(x\mp\frac{1}{2}\varepsilon\right)\right|0\right\rangle = \gamma^{\alpha}\Phi_{\alpha}(x,\varepsilon), \quad (6.8)$$

where $\Phi_{\alpha}(x,\varepsilon)$ is an arbitrary function, and vanishes again between the γ matrices $\gamma_{\mu}\ldots\gamma^{\mu}$.

Substituting (6.7) in (6.5) we obtain

$$j_\mu(x)j^\mu(x) = :\bar\psi(x)\gamma_\mu\psi(x)::\bar\psi(x)\gamma_\mu\psi(x): \\ = \lim_{\varepsilon\to 0} :\bar\psi\left(x+\frac{1}{2}\varepsilon\right)\gamma_\mu\psi\left(x+\frac{1}{2}\varepsilon\right): \\ \times :\bar\psi\left(x-\frac{1}{2}\varepsilon\right)\gamma_\mu\psi\left(x-\frac{1}{2}\varepsilon\right): \\ = \lim_{\varepsilon\to 0} :\bar\psi\left(x+\frac{1}{2}\varepsilon\right)\gamma_\mu\psi\left(x+\frac{1}{2}\varepsilon\right) \\ \times \bar\psi\left(x-\frac{1}{2}\varepsilon\right)\gamma_\mu\psi\left(x-\frac{1}{2}\varepsilon\right):. \quad (6.9)$$

Therefore, in the trivial vacuum we get the relation

$$j_\mu(x,t)j^\mu(x,t) \\ =:\bar\psi(x,t)\gamma_\mu\psi(x,t)::\bar\psi(x,t)\gamma_\mu\psi(x,t): \\ =:\bar\psi(x,t)\gamma_\mu\psi(x,t)\bar\psi(x,t)\gamma_\mu\psi(x,t):, \quad (6.10)$$

where we have taken the limit $\varepsilon\to 0$ and come back to the notation $x\to(x,t)$.

Due to the relation (6.10), for a system of massless fermions self-coupled in the chiral symmetric phase with a trivial chirally invariant vacuum, the Lagrangian (6.1) acquires the form

$$\mathcal{L}(x,t) = :\bar\psi(x,t)\mathrm{i}\gamma^\mu\partial_\mu\psi(x,t): \\ -\frac{1}{2}g:\bar\psi(x,t)\gamma_\mu\psi(x,t)::\bar\psi(x,t)\gamma^\mu\psi(x,t): \\ = :\bar\psi(x,t)\mathrm{i}\gamma^\mu\partial_\mu\psi(x,t): \\ -\frac{1}{2}g j_\mu(x,t)j^\mu(x,t). \quad (6.11)$$

Now let us show that the fermion system described by the Lagrangian (6.11) is unstable under chiral symmetry breaking. In order to reach this aim we rewrite the Lagrangian in an equivalent form:

$$\mathcal{L}(x,t) = :\bar\psi(x,t)(\mathrm{i}\gamma^\mu\partial_\mu - M)\psi(x,t): \\ + M:\bar\psi(x,t)\psi(x,t): \quad (6.12) \\ -\frac{1}{2}g:\bar\psi(x,t)\gamma_\mu\psi(x,t)::\bar\psi(x,t)\gamma^\mu\psi(x,t):$$

and normal order the interaction term at the scale M

$$:\bar\psi(x,t)\gamma_\mu\psi(x,t)::\bar\psi(x,t)\gamma^\mu\psi(x,t): \quad (6.13) \\ =:\bar\psi(x,t)\gamma_\mu\psi(x,t)\bar\psi(x,t)\gamma^\mu\psi(x,t): \\ +2:\bar\psi(x,t)\gamma_\mu\langle 0|\psi(x,t)\bar\psi(x,t)|0\rangle\gamma^\mu\psi(x,t): \\ -\mathrm{tr}\Big\{\gamma_\mu\langle 0|\psi(x,t)\bar\psi(x,t)|0\rangle\gamma^\mu\langle 0|\psi(x,t)\bar\psi(x,t)|0\rangle\Big\}.$$

The vacuum expectation value in the r.h.s. of (6.13) calculated in the one-fermion loop approximation for massive fermions of mass M reads

$$\langle 0|\psi(x,t)\bar\psi(x,t)|0\rangle \to \left\langle 0\left|\psi\left(x\pm\frac{1}{2}\varepsilon\right)\bar\psi\left(x\mp\frac{1}{2}\varepsilon\right)\right|0\right\rangle \\ = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{4\pi}\frac{\gamma^0E_p-\gamma^1p+M}{E_p}\mathrm{e}^{\mp\mathrm{i}(E_p\varepsilon^0-p\varepsilon^1)} \\ = \pm\frac{\hat\varepsilon}{\sqrt{\varepsilon^2}}\frac{M}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\varphi\cosh\varphi\,\mathrm{e}^{-\mathrm{i}M\sqrt{\varepsilon^2}\cosh\varphi} \\ +\frac{M}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\varphi\,\mathrm{e}^{-\mathrm{i}M\sqrt{\varepsilon^2}\cosh\varphi} \\ = \pm\frac{\hat\varepsilon}{\sqrt{\varepsilon^2}}\frac{M}{2\pi}K_1\left(\mathrm{i}M\sqrt{\varepsilon^2}\right)+\frac{M}{2\pi}K_0\left(\mathrm{i}M\sqrt{\varepsilon^2}\right), \quad (6.14)$$

where $E_p=(p^2+M^2)^{1/2}$ and $K_1(z)$ and $K_0(z)$ are McDonald's functions. In the r.h.s. of (6.13) the contribution of the first term proportional to $\hat\varepsilon$ vanishes due to the identities $\gamma_\mu\hat\varepsilon\gamma^\mu=0$ and $\mathrm{tr}\{\hat\varepsilon\}=0$. A non-zero contribution comes only from the second term that coincides with the causal Green function of the scalar field with a mass M and can be regularized in the limit $\varepsilon\to 0$ by the cut-off Λ:

$$\frac{M}{2\pi}K_0(\mathrm{i}M\sqrt{\varepsilon^2}) = M\int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\frac{\mathrm{e}^{\mp\mathrm{i}p\cdot\varepsilon}}{M^2-p^2-\mathrm{i}0} \\ \stackrel{\varepsilon\to 0}{\longrightarrow}\int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\frac{M}{M^2-p^2-\mathrm{i}0} \\ = \frac{M}{4\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right). \quad (6.15)$$

Substituting (6.13) with the vacuum expectation value (6.14) in (6.12) we obtain

$$\mathcal{L}(x,t) = :\bar\psi(x,t)(\mathrm{i}\gamma^\mu\partial_\mu - M)\psi(x,t): \quad (6.16) \\ -\frac{1}{2}g:\bar\psi(x,t)\gamma_\mu\psi(x,t)\bar\psi(x,t)\gamma^\mu\psi(x,t): \\ +\left[M-g\frac{M}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right)\right]:\bar\psi(x,t)\psi(x,t):,$$

where we have omitted an insignificant constant. Self-consistency of the approach demands the relation

$$M-g\frac{M}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right)=0,$$

that is, our gap equation (1.14). This results in the Lagrangian

$$\mathcal{L}(x,t) = :\bar\psi(x,t)(\mathrm{i}\gamma^\mu\partial_\mu - M)\psi(x,t): \quad (6.17) \\ -\frac{1}{2}g:\bar\psi(x,t)\gamma_\mu\psi(x,t)\bar\psi(x,t)\gamma^\mu\psi(x,t):.$$

For $M\neq 0$ the Lagrangian (6.16) describes a system of fermions with mass M in the chirally broken phase. We conclude that for an attractive two-body interaction the vacuum expectation values in (6.5) lead to an instability of the perturbative vacuum. In the next subsection we show that a stable non-perturbative vacuum can be determined within the BCS formalism.

6.2 The massless Thirring model in the formalism of the BCS theory of superconductivity. Chiral symmetry breaking

We discuss the phenomenon of spontaneous breaking of chiral symmetry in the massless Thirring model from the point of view of the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity [23]. We show that the energy density of the non-perturbative vacuum acquires a minimum, when the dynamical mass M of fermions satisfies the gap equation (1.14).

The Hamiltonian of the massless Thirring model is equal to

$$\mathcal{H}(x,t) = -:\bar{\psi}(x,t)\mathrm{i}\gamma^1\frac{\partial}{\partial x}\psi(x,t): + \frac{1}{2}g:\bar{\psi}(x,t)\gamma_\mu\psi(x,t)\bar{\psi}(x,t)\gamma^\mu\psi(x,t):. \tag{6.18}$$

In terms of the components of the ψ field, a column matrix $\psi(x,t) = (\psi_1(x,t), \psi_2(x,t))$, the Hamiltonian (6.18) reads

$$\begin{aligned}\mathcal{H}(x,t) &= -:\psi_1^\dagger(x,t)\mathrm{i}\frac{\partial}{\partial x}\psi_1(x,t): \\ &\quad + :\psi_2^\dagger(x,t)\mathrm{i}\frac{\partial}{\partial x}\psi_2(x,t): \\ &\quad + 2g:\psi_1^\dagger(x,t)\psi_1(x,t)\psi_2^\dagger(x,t)\psi_2(x,t):.\end{aligned} \tag{6.19}$$

For the further evaluation it is convenient to embed the fermion system into a finite volume L. For periodical conditions $\psi(0,t) = \psi(L,t)$ the expansion of $\psi(x,t)$ into plane waves reads (see Appendix D):

$$\psi(x,t) = \sum_{p^1}\frac{1}{\sqrt{2p^0L}}\Big[u(p^0,p^1)a(p^1)\mathrm{e}^{-\mathrm{i}p^0t+\mathrm{i}p^1x} + v(p^0,p^1)b^\dagger(p^1)\mathrm{e}^{\mathrm{i}p^0t-\mathrm{i}p^1x}\Big]. \tag{6.20}$$

The creation and annihilation operators are dimensionless and obey the anti-commutation relations

$$\begin{aligned}\{a(p^1),a^\dagger(q^1)\} &= \{b(p^1),b^\dagger(q^1)\} = \delta_{p^1q^1}, \\ \{a(p^1),a(q^1)\} &= \{a^\dagger(p^1),a^\dagger(q^1)\} = \{b(p^1),b(q^1)\} \\ &= \{b^\dagger(p^1),b^\dagger(q^1)\} = 0.\end{aligned} \tag{6.21}$$

They are related to the annihilation operators of fermions $A(p^1)$ and anti-fermions $B(p^1)$ with mass M by the Bogoliubov transformation [18,24]

$$\begin{aligned}A(p^1) &= u_{p^1}a(p^1) - v_{p^1}b^\dagger(-p^1), \\ B(p^1) &= u_{p^1}b(p^1) - v_{p^1}a^\dagger(-p^1).\end{aligned} \tag{6.22}$$

The coefficient functions u_{p^1} and v_{p^1} are equal to [18,22,24]:

$$\begin{aligned}u_{p^1} &= \sqrt{\frac{1}{2}\left(1+\frac{|p^1|}{\sqrt{(p^1)^2+M^2}}\right)}, \\ v_{p^1} &= \varepsilon(p^1)\sqrt{\frac{1}{2}\left(1-\frac{|p^1|}{\sqrt{(p^1)^2+M^2}}\right)},\end{aligned} \tag{6.23}$$

where $\varepsilon(p^1)$ is a sign function, and obey the normalization condition

$$u_{p^1}^2 + v_{p^1}^2 = 1. \tag{6.24}$$

The wave function of the non-perturbative vacuum $|\Omega\rangle$ we take in the BCS form [23]:

$$|\Omega\rangle = \prod_{k^1}[u_{k^1} + v_{k^1}a^\dagger(k^1)b^\dagger(-k^1)]|0\rangle, \tag{6.25}$$

where $|0\rangle$ is a perturbative, chiral symmetric vacuum. The wave function $|\Omega\rangle$ satisfies the equations

$$A(p^1)|\Omega\rangle = B(p^1)|\Omega\rangle = 0 \tag{6.26}$$

and is invariant under parity transformation:

$$\begin{aligned}\mathcal{P}\psi(x,t)\mathcal{P}^\dagger &= \gamma^0\psi(-x,t) \\ \Longrightarrow \mathcal{P}\psi_1(x,t)\mathcal{P}^\dagger &= \psi_2(-x,t), \\ \mathcal{P}\psi_2(x,t)\mathcal{P}^\dagger &= \psi_1(-x,t), \\ \mathcal{P}a^\dagger(k^1)\mathcal{P}^\dagger &= +a^\dagger(-k^1), \\ \mathcal{P}b^\dagger(k^1)\mathcal{P}^\dagger &= -b^\dagger(-k^1),\end{aligned} \tag{6.27}$$

where we have dropped insignificant phase factors.

Due to the relation (6.24) the wave function of the non-perturbative vacuum is normalized to unity $\langle\Omega|\Omega\rangle = 1$. The coefficient functions u_{k^1} and v_{k^1} depend explicitly on the dynamical M which we treat as a variational parameter.

The energy of the ground state is equal to [23]

$$\begin{aligned}E(M) &= \int_{-\infty}^{\infty}\mathrm{d}x\langle\Omega|\mathcal{H}(x,t)|\Omega\rangle \\ &= 4\sum_{p^1>0}p^1v_{p^1}^2 - \frac{8g}{L}\left[\left(\sum_{p^1>0}v_{p^1}u_{p^1}\right)^2 + \frac{1}{2}\sum_{p^1>0}v_{p^1}^4\right].\end{aligned} \tag{6.28}$$

The energy density we define by

$$\begin{aligned}\mathcal{E}(M) &= \lim_{L\to\infty}\frac{E(M)}{L} = 4\int_0^\infty\frac{\mathrm{d}p^1}{2\pi}p^1v^2(p^1) \\ &\quad - 8g\left[\int_0^\infty\frac{\mathrm{d}p^1}{2\pi}v(p^1)u(p^1)\right]^2.\end{aligned} \tag{6.29}$$

Substituting (6.23) in (6.29) we express the energy density as a function of the variational parameter M:

$$\begin{aligned}\mathcal{E}(M) &= 2\int_0^\infty\frac{\mathrm{d}p^1}{2\pi}p^1\left(1-\frac{p^1}{\sqrt{(p^1)^2+M^2}}\right) \\ &\quad - 2g\left[\int_0^\infty\frac{\mathrm{d}p^1}{2\pi}\frac{M}{\sqrt{(p^1)^2+M^2}}\right]^2.\end{aligned} \tag{6.30}$$

The minimum of the energy density is defined by

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}(M)}{\mathrm{d}M} &= \left[M - 2g\int_0^\infty\frac{\mathrm{d}p^1}{2\pi}\frac{M}{\sqrt{(p^1)^2+M^2}}\right] \\ &\quad\times\int_0^\infty\frac{\mathrm{d}p^1}{\pi}\frac{(p^1)^2}{((p^1)^2+M^2)^{3/2}} = 0.\end{aligned} \tag{6.31}$$

This yields the BCS-like gap equation

$$M = 2g\int_0^{\infty}\frac{dp^1}{2\pi}\frac{M}{\sqrt{(p^1)^2+M^2}}. \tag{6.32}$$

Calculating the second derivative of $\mathcal{E}(M)$ with respect to M one can show that the BCS gap equation describes the minimum of the energy density (6.30) at $M \neq 0$. Using the obvious relation

$$\int_0^{\infty}\frac{dp^1}{2\pi}\frac{M}{\sqrt{(p^1)^2+M^2}} = \int\frac{d^2p}{(2\pi)^2 i}\frac{M}{M^2-p^2-i0} = \frac{M}{4\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right) \tag{6.33}$$

we obtain the gap equation (6.32) in the form (1.14).

Using the relations between momentum integrals (6.33) and

$$2\int_0^{\infty}\frac{dp^1}{2\pi}p^1\left(1-\frac{p^1}{\sqrt{(p^1)^2+M^2}}\right) = -\int\frac{d^2p}{(2\pi)^2 i}\ln(M^2-p^2-i0) + \int\frac{d^2p}{(2\pi)^2 i}\frac{2M^2}{M^2-p^2-i0}+C, \tag{6.34}$$

where C is a infinite constant independent of M, and the gap equation (1.14) the energy density $\mathcal{E}(M)$ can be transformed into the form

$$\mathcal{E}(M) = \frac{1}{4\pi}\left[M^2\ln\frac{M^2}{\Lambda^2}-(\Lambda^2+M^2)\ln\left(1+\frac{M^2}{\Lambda^2}\right) + \frac{2\pi}{g}M^2\right]+C', \tag{6.35}$$

where C' is an infinite constant independent on M.

Using the normalization $\mathcal{E}(0)=0$ we obtain

$$\mathcal{E}(M) = \frac{1}{4\pi}\left[M^2\ln\frac{M^2}{\Lambda^2}-(\Lambda^2+M^2)\ln\left(1+\frac{M^2}{\Lambda^2}\right) + \frac{2\pi}{g}M^2\right]. \tag{6.36}$$

This is evidence for the complete agreement of the energy density of the BCS-like ground state with the effective potential $V[M]$ given by (3.21) at $\bar{\rho}(x)=M$:

$$\mathcal{E}(M) = V[M]. \tag{6.37}$$

Thus, we have shown explicitly that the chirally broken phase in the massless Thirring model is a superconducting phase of the BCS type with the BCS non-perturbative superconducting vacuum.

Since the energy density $\mathcal{E}(M)$ has a maximum at $M=0$, it is obvious that for Thirring fermions the chirally broken phase is energetically preferred with respect to the chiral symmetric phase.

One can show that fixing $M=$ const and letting $\Lambda\to\infty$ the effective potential and the energy density tend to a finite limit $\mathcal{E}(M)=V[M]\to -M^2/4\pi$. This means that the energy spectrum of the ground state of the massless Thirring model is restricted from below for $\Lambda\to\infty$.

Now let investigate chiral properties of the wave function of the ground state (6.25) under chiral rotations of the massless Thirring fermion fields:

$$\psi(x,t)\to\psi'(x,t) = e^{i\gamma^5\alpha_A}\psi(x,t),$$
$$\bar{\psi}(x,t)\to\bar{\psi}'(x,t) = \bar{\psi}(x,t)e^{i\gamma^5\alpha_A}. \tag{6.38}$$

The operators of annihilation and creation transform under chiral rotations (6.38) as follows [18,24]:

$$a(k^1)\to a'(k^1) = e^{+i\alpha_A\varepsilon(k^1)}a(k^1),$$
$$b(k^1)\to b'(k^1) = e^{-i\alpha_A\varepsilon(k^1)}b(k^1),$$
$$a^{\dagger}(k^1)\to a^{\dagger\prime}(k^1) = e^{-i\alpha_A\varepsilon(k^1)}a^{\dagger}(k^1),$$
$$b^{\dagger}(k^1)\to b^{\dagger\prime}(k^1) = e^{+i\alpha_A\varepsilon(k^1)}b^{\dagger}(k^1). \tag{6.39}$$

The wave function of the non-perturbative vacuum of the massless Thirring model (6.25) is not invariant under chiral rotations (6.38) and (6.39) [18,24]:

$$|\Omega\rangle\to|\Omega;\alpha_A\rangle = \prod_{k^1}[u_{k^1}+v_{k^1}e^{-2i\alpha_A\varepsilon(k^1)}a^{\dagger}(k^1)b^{\dagger}(-k^1)]|0\rangle, \tag{6.40}$$

One can show that the energy density $\mathcal{E}(M)$ does not depend on the phase of the function v_{k^1}.

The scalar product $\langle\alpha'_A;\Omega|\Omega;\alpha_A\rangle$ of the wave function for $\alpha'_A\neq\alpha_A$ is equal to [18,24]

$$\langle\alpha'_A;\Omega|\Omega;\alpha_A\rangle = \exp\left\{\frac{L}{2\pi}\int_0^{\infty}dk^1 \times\ln\left[1-\sin^2(\alpha'_A-\alpha_A)\frac{M^2}{M^2+(k^1)^2}\right]\right\}. \tag{6.41}$$

In the limit $L\to\infty$ the wave functions $|\Omega;\alpha'_A\rangle$ and $|\Omega;\alpha_A\rangle$ are orthogonal for $\alpha'_A\neq\alpha_A$ [18,24].

The wave function (6.25) is invariant under parity transformation: $\mathcal{P}|\Omega\rangle=|\Omega\rangle$. The operator

$$\mathcal{O}_+ = 2\sum_{p^1}\varepsilon(p^1)b^{\dagger}(-p^1)a^{\dagger}(p^1) \tag{6.42}$$

is invariant under parity transformations (6.27): $\mathcal{P}\mathcal{O}_+\mathcal{P}^{\dagger}=\mathcal{O}_+$. Its vacuum expectation value of the operator $\mathcal{O}_+$ per unit volume can be identified with the order parameter for the massless Thirring model in the BCS formalism

$$\langle\mathcal{O}_+\rangle = \frac{1}{L}\langle\Omega|\mathcal{O}_+|\Omega\rangle = -\frac{2}{L}\sum_{p^1}u_{p^1}v_{p^1} = -\int_{-\infty}^{\infty}\frac{dp^1}{2\pi}\frac{M}{\sqrt{M^2+(p^1)^2}} = -\frac{M}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right), \tag{6.43}$$

where we have used (6.33). The expectation value $\langle\mathcal{O}_+\rangle$ is the fermion condensate

$$\langle\Omega| : \bar\psi(0)\psi(0) : |\Omega\rangle = \langle\mathcal{O}_+\rangle = -\frac{M}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right), \quad (6.44)$$

where we have used $\langle\mathcal{O}_+\rangle = \langle\mathcal{O}^\dagger_+\rangle$.

Thus, we have shown that the ground state of the massless Thirring model possesses all properties of a BCS type superconducting vacuum, and our results obtained by means of the path integral approach are fully reproducible within the BCS formalism.

6.3 Equations of motion and integral of motion

Now let us turn to the analysis of the equations of motion. Using the Lagrangian (6.11) we derive the equations of motion

$$\begin{aligned} \mathrm{i}\gamma^\mu\partial_\mu\psi(x,t) &= g j^\mu(x,t)\gamma_\mu\psi(x,t),\\ -\mathrm{i}\partial_\mu\bar\psi(x,t)\gamma^\mu &= g\bar\psi(x,t)\gamma_\mu j^\mu(x,t). \end{aligned} \quad (6.45)$$

Due to the peculiarity of $1+1$-dimensional quantum field theories of fermion fields [32] these equations are equivalent to

$$\begin{aligned} \mathrm{i}\partial_\mu\psi(x,t) &= a j_\mu(x,t)\psi(x,t)\\ &\quad + b\varepsilon_{\mu\nu}j^\nu(x,t)\gamma^5\psi(x,t),\\ -\mathrm{i}\partial_\mu\bar\psi(x,t) &= a\bar\psi(x,t)j_\mu(x,t)\\ &\quad + b\bar\psi(x,t)\gamma^5 j^\nu(x,t)\varepsilon_{\nu\mu}, \end{aligned} \quad (6.46)$$

where the parameters a and b are constrained by the condition $a+b=g$[6]. Multiplying the equations (6.46) by γ^μ and summing over $\mu=0,1$ we end up with the equations of motion (6.45).

From the equations of motion (6.46) we can get a very important information about the evolution of fermions in the massless Thirring model. For this we write (6.46) in the form

$$\begin{aligned} -\partial_\mu[\bar\psi(x,t)\psi(x,t)] &= 2b\varepsilon_{\mu\nu}j^\nu(x,t)\\ &\quad\times[\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)],\\ \partial_\mu[\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)] &= 2b\varepsilon_{\mu\nu}j^\nu(x,t)\\ &\quad\times[\bar\psi(x,t)\psi(x,t)], \end{aligned} \quad (6.47)$$

exclude $2b\varepsilon_{\mu\nu}j^\nu(x,t)$ and arrive at the equation

$$\partial_\alpha([\bar\psi(x,t)\psi(x,t)]^2 + [\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)]^2) = 0. \quad (6.48)$$

This means that the quantity

$$[\bar\psi(x,t)\psi(x,t)]^2 + [\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)]^2 = \text{const.} \quad (6.49)$$

is an integral of motion. Using the equations of motion (3.24) and going to the polar representation $\sigma(x,t) = \rho(x,t)\cos\beta\vartheta(x,t)$ and $\varphi(x,t) = \rho(x,t)\sin\beta\vartheta(x,t)$ we get

$$\begin{aligned} &[\bar\psi(x,t)\psi(x,t)]^2 + [\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)]^2\\ &\quad = \frac{\rho^2(x,t)}{g^2} = \text{const.} \end{aligned} \quad (6.50)$$

[6] Below we show that $a-b=1/c$ where c is the Schwinger term [33]

Substituting (6.50) in (6.48) we obtain the equation of motion for the ρ field

$$\partial_\alpha\rho(x,t) = 0. \quad (6.51)$$

This gives $\rho(x,t)=\rho(0)$. The value of $\rho(0)$ can be fixed by noticing that for $\partial_\alpha\rho(x,t)=0$ the effective Lagrangian of the ρ field is defined by the effective potential (3.21), $\mathcal{L}_{\rm eff}[\rho(x,t)] = -V[\rho(x,t)]$. In this case the equation of motion for the ρ field reads

$$\frac{\delta\mathcal{L}_{\rm eff}[\rho(x,t)]}{\delta\rho(x,t)} = -\frac{\delta V[\rho(x,t)]}{\delta\rho(x,t)} = 0 \quad (6.52)$$

and coincides with the extremum condition (3.22) with the solution $\rho(x,t)=\rho_0=M$. This implies that the solution of (6.51) which is the equation of motion for the ρ field should be $\rho(x,t)=\rho(0)=M$.

Since the terms $[\bar\psi(x,t)\psi(x,t)]^2$ and $[\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)]^2$ are positively defined, we predict $\rho(x,t)\neq 0$. This testifies the stability of the chirally broken phase during the evolution of the fermion system described by the massless Thirring model evolving from the symmetry broken phase.

Setting $\rho(x,t)=M+\tilde\rho(x,t)$, where the $\tilde\rho$ field describes fluctuations of the ρ field around the minimum of the effective potential, the equation of motion (6.51) reduces to the form

$$\partial_\alpha\tilde\rho(x,t) = 0. \quad (6.53)$$

In Appendix B we show that the $\tilde\rho$ field decouples from the system. The direct consequence of this decoupling is $\tilde\rho(x,t)=0$ as solution of (6.53).

An example of classical solutions of the equations of motion (6.45) and (6.46) for fermions evolving in the chirally broken phase and obeying the integral of motion

$$[\bar\psi(x,t)\psi(x,t)]^2 + [\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)]^2 = \frac{M^2}{g^2} \quad (6.54)$$

is given by the ansatz

$$\psi(x,t) = \sqrt{-\frac{M}{2g}}\begin{pmatrix} \mathrm{e}^{+\omega/2}\mathrm{e}^{-\mathrm{i}\xi(x,t)}\\ \mathrm{e}^{-\omega/2}\mathrm{e}^{+\mathrm{i}\eta(x,t)}\end{pmatrix}, \quad (6.55)$$

where ω is a arbitrary real parameter and $\xi(x,t)+\eta(x,t) = \beta\vartheta(x,t)$. In terms of (6.55) the scalar $\bar\psi(x,t)\psi(x,t)$and pseudoscalar $\bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t)$ densities read

$$\begin{aligned} \bar\psi(x,t)\psi(x,t) &= -\frac{M}{g}\cos\beta\vartheta(x,t),\\ \bar\psi(x,t)\mathrm{i}\gamma^5\psi(x,t) &= -\frac{M}{g}\sin\beta\vartheta(x,t). \end{aligned} \quad (6.56)$$

The functions $\xi(x,t)$ and $\eta(x,t)$ are found in Appendix C.

6.4 Current algebra and the Schwinger term

The canonical conjugate momentum of the ψ field is defined by

$$\pi(x,t) = \frac{\delta\mathcal{L}(x,t)}{\delta\partial_0\psi(x,t)} = \mathrm{i}\psi^\dagger(x,t). \quad (6.57)$$

The canonical anti-commutation relations read therefore

$$\{\psi(x,t),\pi(y,t)\} = \mathrm{i}\delta(x-y) \to \{\psi(x,t),\psi^\dagger(y,t)\} = \delta(x-y). \quad (6.58)$$

Using the canonical anti-commutation relations (6.58) one can see that the equal-time commutation relations $[j_\mu(x,t), j_\nu(y,t)]$ vanish for each choice of μ and ν [30]. However, according to Schwinger [29] the equal-time commutation relations $[j_\mu(x,t), j_\nu(y,t)]$ should read

$$\begin{aligned}[][j_0(x,t), j_0(y,t)] &= 0,\\ [j_0(x,t), j_1(y,t)] &= \mathrm{i}c\frac{\partial}{\partial x}\delta(x-y),\\ [j_1(x,t), j_1(y,t)] &= 0,\end{aligned} \quad (6.59)$$

where c is the Schwinger term [33].

In the massless Thirring model the Schwinger term c has been calculated by Sommerfield [34], with the result $c = 1/\pi$. Now let us show that this result is due to the triviality of the vacuum in the chiral symmetric phase of the massless Thirring model. We will get another value for c in the spontaneously broken phase.

For this aim we evaluate the vacuum expectation value of the equal-time commutation relation $[j_0(x,t), j_1(y,t)]$. In the one-fermion loop approximation for free fermions with mass M, sufficient as has been shown above for the description of the dynamics of a fermion system in the chirally broken phase, we get

$$\begin{aligned}&\langle 0|[j_0(x,t), j_1(y,t)]|0\rangle\\ &= \langle 0|[:\bar\psi(x,t)\gamma_0\psi(x,t):, :\bar\psi(y,t)\gamma_1\psi(y,t):]|0\rangle\\ &= -\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\mathrm{e}^{\mathrm{i}k(x-y)}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{4\pi}\left[\frac{k-q}{\sqrt{(k-q)^2+M^2}}\right.\\ &\quad\left.+\frac{k+q}{\sqrt{(k+q)^2+M^2}}\right].\end{aligned} \quad (6.60)$$

For the integration over q,

$$\begin{aligned}&\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{4\pi}\left[\frac{k-q}{\sqrt{(k-q)^2+M^2}}+\frac{k+q}{\sqrt{(k+q)^2+M^2}}\right]\\ &= \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{4\pi}\left[\frac{q+k}{\sqrt{(q+k)^2+M^2}}-\frac{q-k}{\sqrt{(q-k)^2+M^2}}\right]\\ &= \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{4\pi}\int\limits_{-1}^{1}\mathrm{d}\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{q+k\alpha}{\sqrt{(q+k\alpha)^2+M^2}}\right]\\ &= \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{4\pi}\int\limits_{-1}^{1}\mathrm{d}\alpha\frac{kM^2}{((q+k\alpha)^2+M^2)^{3/2}}\\ &= \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{4\pi}\int\limits_{-1}^{1}\mathrm{d}\alpha\frac{kM^2}{(q^2+M^2)^{3/2}}\\ &= k\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}\frac{M^2}{(q^2+M^2)^{3/2}}\\ &= k\int\frac{\mathrm{d}^2q}{\pi^2\mathrm{i}}\frac{M^2}{(M^2-q^2-\mathrm{i}0)^2}\\ &= k\frac{1}{\pi}\frac{\varLambda^2}{M^2+\varLambda^2},\end{aligned} \quad (6.61)$$

we used the relation

$$\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}\frac{M^2}{(q^2+M^2)^{3/2}} = \int\frac{\mathrm{d}^2q}{\pi^2\mathrm{i}}\frac{M^2}{(M^2-q^2-\mathrm{i}0)^2}. \quad (6.62)$$

The r.h.s. of (6.62) represented in relativistic invariant form is regularized according to our approach to the evaluation of the effective Lagrangians (3.35) and (5.14).

Substituting (6.61) in (6.60) we obtain

$$\begin{aligned}&\langle 0|[j_0(x,t), j_1(y,t)]|0\rangle\\ &= \langle 0|[:\bar\psi(x,t)\gamma_0\psi(x,t):, :\bar\psi(y,t)\gamma_1\psi(y,t):]|0\rangle\\ &= \mathrm{i}\frac{1}{\pi}\frac{\varLambda^2}{M^2+\varLambda^2}\frac{\partial}{\partial x}\delta(x-y).\end{aligned} \quad (6.63)$$

Following Schwinger [33] and Sommerfield [34] we write down the equal-time commutation relation

$$[j_0(x,t), j_1(y,t)] = \mathrm{i}\frac{1}{\pi}\frac{\varLambda^2}{M^2+\varLambda^2}\frac{\partial}{\partial x}\delta(x-y). \quad (6.64)$$

When matching (6.64) with (6.59) we derive the Schwinger term

$$c = \frac{1}{\pi}\frac{\varLambda^2}{M^2+\varLambda^2}. \quad (6.65)$$

For the chiral symmetric phase with $M = 0$ the Schwinger term is equal to that obtained by Sommerfield $c = 1/\pi$ [34], whereas in the chirally broken phase, when M is defined by (1.15), we get the new value

$$c = \frac{1}{\pi}(1-\mathrm{e}^{-2\pi/g}). \quad (6.66)$$

This agrees with the dependence of β^2 on g given by (4.5).

6.5 Energy-momentum tensor and its Sugawara form

The energy-momentum tensor $\theta_{\mu\nu}(x,t)$ of the fermion system described by the Lagrangian $\mathcal{L}(x,t)$ is defined by

$$\begin{aligned}\theta_{\mu\nu}(x,t) &= \frac{1}{2}:\partial_\mu\bar\psi(x,t)\frac{\delta\mathcal{L}(x,t)}{\delta\partial^\nu\bar\psi(,t)}:\\ &\quad+\frac{1}{2}:\frac{\delta\mathcal{L}(x,t)}{\delta\partial^\mu\psi(x,t)}\partial_\nu\psi(x,t):+(\mu\leftrightarrow\nu)\\ &\quad-\mathcal{L}(x,t)g_{\mu\nu}.\end{aligned} \quad (6.67)$$

For the massless Thirring model with the Lagrangian (6.1) the energy-momentum tensor $\theta_{\mu\nu}(x,t)$ reads

$$\begin{aligned}\theta_{\mu\nu}(x,t) &= \frac{1}{4} : \bar{\psi}(x,t)\mathrm{i}\gamma_\mu\partial_\nu\psi(x,t) : \\ &+ \frac{1}{4} : \bar{\psi}(x,t)\mathrm{i}\gamma_\nu\partial_\mu\psi(x,t) : \\ &- \frac{1}{4} : \partial_\mu\bar{\psi}(x,t)\mathrm{i}\gamma_\nu\psi(x,t) : \\ &- \frac{1}{4} : \partial_\nu\bar{\psi}(x,t)\mathrm{i}\gamma_\mu\psi(x,t) : \\ &- g_{\mu\nu} : \Big[\bar{\psi}(x,t)\mathrm{i}\gamma^\alpha\partial_\alpha\psi(x,t) \\ &- \frac{1}{2}g\bar{\psi}(x,t)\gamma_\alpha\psi(x,t)\bar{\psi}(x,t)\gamma^\alpha\psi(x,t)\Big] : .\end{aligned} \tag{6.68}$$

As has been pointed out by Callan, Dashen and Sharp [30] the components of the energy-momentum tensor $\theta_{\mu\nu}(x,t)$ in the massless Thirring model obey the same equal-time commutation relations as the components of the energy-momentum tensor $\theta^S_{\mu\nu}(x,t)$ defined in terms of the vector current $j_\mu(x,t)$ only:

$$\begin{aligned}\theta^S_{\mu\nu}(x,t) &= \frac{1}{2c}\Big[j_\mu(x,t)j_\nu(x,t) + j_\nu(x,t)j_\mu(x,t) \\ &- g_{\mu\nu}j_\alpha(x,t)j^\alpha(x,t)\Big],\end{aligned} \tag{6.69}$$

where c is the Schwinger term. A quantum field theory with currents as quantum variables and an energy-momentum tensor of the kind (6.69) has been considered by Sugawara [31].

A direct reduction of the energy-momentum tensor (6.68) to Sugawara's form (6.69) can be carried out using the equations of motion (6.46) as shown by Sommerfield [32]. Substituting (6.46) in (6.68) with the Lagrangian defined by (6.11) we arrive at the expression:

$$\begin{aligned}\theta_{\mu\nu}(x,t) &= \frac{1}{2}(a-b)\Big[j_\mu(x,t)j_\nu(x,t) \\ &+ j_\nu(x,t)j_\mu(x,t) - g_{\mu\nu}j_\alpha(x,t)j^\alpha(x,t)\Big],\end{aligned} \tag{6.70}$$

When matching (6.70) with (6.69) we infer that $a-b=1/c$. In the chirally broken phase the Schwinger term is defined by (6.66).

7 The fermion number as a topological charge of the SG model

The topological properties of the SG model are characterized by the topological current $\mathcal{J}^\mu(x)$ [35]:

$$\mathcal{J}^\mu(x,t) = \frac{\beta}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\vartheta(x,t). \tag{7.1}$$

The spatial integral of its time-component, the topological charge,

$$\begin{aligned}q &= \int\limits_{-\infty}^{\infty}\mathrm{d}x\mathcal{J}^0(x,t) = \frac{\beta}{2\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}x\frac{\partial}{\partial x}\vartheta(x,t) \\ &= \frac{\beta}{2\pi}[\vartheta(\infty)-\vartheta(-\infty)],\end{aligned} \tag{7.2}$$

is conserved irrespective of the equations of motion and is integer valued [35]. The field $\mathrm{e}^{\mathrm{i}\beta\vartheta(x,t)}$, where $\vartheta(x,t)$ is a solution of the equation of motion [1],

$$\Box\vartheta(x,t) + \frac{\alpha}{\beta}\sin\beta\vartheta(x,t) = 0, \tag{7.3}$$

maps at any time t the real axis $\mathcal{R}^1$ onto the circle $\mathcal{S}^1$ with a winding number equal to the topological charge q [35].

For a solitary wave moving with a velocity u, the one-soliton solution of the SG model [1],

$$\vartheta(x,t) = \frac{4}{\beta}\arctan\left[\exp\left(\sqrt{\alpha}\frac{x-ut}{\sqrt{1-u^2}}\right)\right], \tag{7.4}$$

the topological charge q is equal to unity:

$$q = \frac{2}{\pi}\arctan\left[\exp\left(\sqrt{\alpha}\frac{x-ut}{\sqrt{1-u^2}}\right)\right]\Bigg|_{-\infty}^{\infty} = 1. \tag{7.5}$$

In turn, for the anti-soliton solution, $\bar{\vartheta}(x,t)$, given by [1]

$$\bar{\vartheta}(x,t) = \frac{4}{\beta}\arctan\left[\exp\left(-\sqrt{\alpha}\frac{x-ut}{\sqrt{1-u^2}}\right)\right], \tag{7.6}$$

the topological charge $\bar{q}$ amounts to $\bar{q}=-1$.

We argue that in our approach to the bosonization of the massive Thirring model the topological current (7.1) coincides with the Noether current related to the global $U_{\mathrm{V}}(1)$ symmetry of the massive Thirring model. Since this Noether current is responsible for the conservation of the fermion number in the massive Thirring model, this allows one to identify the topological charge with the fermion number.

For the derivation of the Noether current we write the effective Lagrangian of the bosonized version of the massive Thirring model (5.12) in the form

$$\mathcal{L}_{\mathrm{eff}}(x) = -\mathrm{i}\langle x|\mathrm{tr}\ln(\mathrm{i}\gamma^\mu\partial_\mu + \gamma^\mu A_\mu - M)|x\rangle + \ldots, \tag{7.7}$$

where $A_\mu(x) = \beta\varepsilon_{\mu\nu}\partial^\nu\vartheta(x)/2$ with β defined by (4.5) and included in the definition of the A_μ field to get the correct kinetic term for the ϑ field.

Under infinitesimal local $U_{\mathrm{V}}(1)$ rotations with a parameter $\alpha_{\mathrm{V}}(x)$ the vector field $A_\mu(x)$ transforms as[7]

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha_{\mathrm{V}}(x), \tag{7.8}$$

and the effective Lagrangian (7.7) changes as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}(x) &\to \mathcal{L}_{\mathrm{eff}}[\alpha_{\mathrm{V}}(x)] = \mathcal{L}_{\mathrm{eff}}(x) \\ &+ \mathrm{i}\left\langle x\left|\mathrm{tr}\left\{\frac{1}{M-\mathrm{i}\gamma^\nu\partial_\nu-\gamma^\nu A_\nu}\gamma^\mu\right\}\right|x\right\rangle\partial_\mu\alpha_{\mathrm{V}}(x).\end{aligned} \tag{7.9}$$

[7] Transformations of fermion fields under local $U_{\mathrm{V}}(1)$ rotations with a parameter $\alpha_{\mathrm{V}}(x)$ are defined by (6.2)

The Noether current is defined by [22]

$$J^{\mu}(x) = -\frac{\delta\mathcal{L}_{\rm eff}[\alpha_{\rm V}(x)]}{\delta\partial_{\mu}\alpha_{\rm V}(x)} = -{\rm i}\left\langle x\left|{\rm tr}\left\{\frac{1}{M - {\rm i}\gamma^{\nu}\partial_{\nu} - \gamma^{\nu}A_{\nu}}\gamma^{\mu}\right\}\right|x\right\rangle. \quad (7.10)$$

The explicit calculation of the matrix element in the r.h.s. of (7.10) gives

$$J^{\mu}(x) = \frac{1}{2\pi}(1 - {\rm e}^{-2\pi/g})A^{\mu}(x) = \frac{2}{\beta}\varepsilon^{\mu\nu}\partial_{\nu}\vartheta(x), \quad (7.11)$$

where we have used relation (4.5).

Thus, the topological current $\mathcal{J}^{\mu}(x)$ of (7.1) is proportional to the Noether current (7.11) with $\beta^2/4\pi$ as coefficient of proportionality.

Since the topological current coincides with the Noether current related to the $U_{\rm V}(1)$ symmetry of the massive Thirring model, which is responsible for the conservation of the fermion number, the topological charge q has the meaning of the fermion number.

This leads to the conclusion that the solitons of the SG model can be identified with fermions, as conjectured by Skyrme [4].

It is interesting to note that the mass of the soliton [1] $M_{\rm sol} = 8(\alpha)^{1/2}/\beta^2$ can be represented in a form resembling the Gell-Mann–Oakes–Renner low-energy theorem for the mass spectrum of low-lying pseudoscalar mesons [36]

$$M^2_{\rm sol} = \frac{64\alpha}{\beta^4} = -\frac{64}{\beta^2}m\langle\bar{\psi}\psi\rangle + O(m^2), \quad (7.12)$$

where we have used (4.5) and (5.15).

8 Chiral symmetry breaking in the massless Thirring model and the Mermin–Wagner theorem

In this section we would like to show that our approach to the bosonization of the massless Thirring model does not contradict the Mermin–Wagner theorem [25]. According to this theorem there is no spontaneously broken continuous symmetry in two-dimensional quantum field theories. The essence of the Mermin–Wagner theorem can be illustrated by the classical Heisenberg model with a continuous $O(n)$ symmetry, where dynamical variables are unit vectors $\boldsymbol{S}_i$ on a sphere [27]. Following Mermin and Wagner [25] one can show [27] that there is no spontaneous magnetization for $n < 3$. The applicability of the Mermin–Wagner theorem to $1+1$-dimensional quantum field theories has been pointed out in [26,37,38] (see also [27,28]). From a dynamical point of view the Mermin–Wagner theorem states the absence of long-range order in $1+1$-dimensional quantum field theories.

In this connection Coleman [26] argued that in a $1+1$-dimensional quantum field theory of a massless scalar field there are no Goldstone bosons. They accompany, according to Goldstone's theorem [39], the spontaneous breaking of a continuous symmetry. In order to prove this statement Coleman considered a quantum field theory of a massless free scalar field $\varphi(x,t)$ with the Lagrangian

$$\mathcal{L}(x,t) = \frac{1}{2}\partial_{\mu}\varphi(x,t)\partial^{\mu}\varphi(x,t). \quad (8.1)$$

The equation of motion of the φ field reads

$$\Box\varphi(x,t) = 0. \quad (8.2)$$

The Lagrangian (8.1) is invariant under the field translations [40]

$$\varphi(x,t) \to \varphi'(x,t) = \varphi(x,t) - 2\alpha_{\rm A}, \quad (8.3)$$

where $\alpha_{\rm A}$ is an arbitrary parameter. The conserved current associated with these field translations is equal to [40]

$$j_{\mu}(x,t) = \partial_{\mu}\varphi(x,t). \quad (8.4)$$

The total "charge" is defined by the time-component of $j_{\mu}(x,t)$ [40],

$$\lim_{L\to\infty} Q(t) = \lim_{L\to\infty}\int^{L/2}_{-L/2}{\rm d}x\frac{\partial}{\partial t}\varphi(x,t), \quad (8.5)$$

where L is the volume occupied by the system.

It is well known that the spontaneous breaking of a continuous symmetry occurs when the ground state of the system is not invariant under the symmetry group [40]. The ground state of the system described by the Lagrangian (8.1) is not invariant under field translations (8.3) [40]. Thereby, the field-translation symmetry should be spontaneously broken and a Goldstone boson should appear [40].

The absence of Goldstone bosons in the quantum field theory described by the Lagrangian (8.1) Coleman argued by stating the impossibility to construct a massless scalar field operator (see also [27]). This statement has been supported by the analysis of the two-point Wightman function [26,27]:

$$\langle 0|\varphi(x,t)\varphi(0)|0\rangle = \int\frac{{\rm d}^2k}{2\pi}\theta(k^0)\delta(k^2){\rm e}^{{\rm i}k^0t - {\rm i}k^1x} = \frac{1}{2\pi}\int^{\infty}_0\frac{{\rm d}k^1}{k^1}\cos(k^1x){\rm e}^{{\rm i}k^1t}, \quad (8.6)$$

which is defined by *a meaningless infrared divergent integral.* No subtraction procedure can be devised to circumvent this difficulty without spoiling the fundamental properties of field theory, for instance, positivity of the Hilbert space metric. A massless scalar field theory is undefined in a two-dimensional world due to severe infrared divergences [27]. This corresponds to the destruction of long-range order pointed out by Mermin and Wagner [27].

However, in spite of the widely accepted statement by Coleman about the absence of Goldstone bosons in a $1+1$-dimensional quantum field theory described by the Lagrangian (8.1) we argue, nevertheless, that in this theory the field-translation symmetry is spontaneously broken in the sense of the non-invariance of the ground state under transformations (8.3) and Goldstone bosons appear.

Indeed, it has been pointed out by Itzykson and Zuber that in the case of the quantum field theory described by the Lagrangian (8.1) the Goldstone boson is the quantum of the φ field itself [40]. Then, the non-invariance of the ground state of the system can be demonstrated by acting with the operator $\exp\{-2\mathrm{i}\alpha_{\mathrm{A}}Q(0)\}$ on the vacuum wave function $|0\rangle$, i.e. $|\alpha_{\mathrm{A}}\rangle = \exp\{-2\mathrm{i}\alpha_{\mathrm{A}}Q(0)\}|0\rangle$ [40].

For the calculation of $|\alpha_{\mathrm{A}}\rangle$ we follow Itzykson and Zuber [40] and use the Fourier decomposition of the massless scalar field $\varphi(x,t)$:

$$\varphi(x,t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k^1}{2\pi}\frac{1}{2k^0} \times \Big[a(k^1)\mathrm{e}^{-\mathrm{i}k^0t+\mathrm{i}k^1x} + a^{\dagger}(k^1)\mathrm{e}^{\mathrm{i}k^0t-\mathrm{i}k^1x}\Big], \tag{8.7}$$

where $k^0 = |k^1|$, then $a(k^1)$ and $a^{\dagger}(k^1)$ are annihilation and creation operators obeying the standard commutation relation

$$[a(k^1), a^{\dagger}(q^1)] = (2\pi)2k^0\delta(k^1 - q^1). \tag{8.8}$$

From (8.5) we obtain the total "charge" operator $Q(0)$ [40]

$$Q(0) = -\frac{\mathrm{i}}{2}[a(0) - a^{\dagger}(0)]. \tag{8.9}$$

Then, we get the wave function $|\alpha_{\mathrm{A}}\rangle$

$$|\alpha_{\mathrm{A}}\rangle = \mathrm{e}^{-2\mathrm{i}\alpha_{\mathrm{A}}Q(0)}|0\rangle = \mathrm{e}^{-\alpha_{\mathrm{A}}[a(0)-a^{\dagger}(0)]}|0\rangle. \tag{8.10}$$

For the subsequent mathematical operations with the wave functions $|\alpha_{\mathrm{A}}\rangle$ it is convenient to use the regularization procedure suggested by Itzykson and Zuber. We define the regularized operator $Q(0)_{\mathrm{R}}$ as follows [40]:

$$Q(0)_{\mathrm{R}} = \lim_{L\to\infty}\int_{-\infty}^{\infty}\mathrm{d}x\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0}\mathrm{e}^{-x^2/L^2}. \tag{8.11}$$

The regularized wave function $|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}}$ reads then

$$\begin{aligned}|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}} &= \mathrm{e}^{-2\mathrm{i}\alpha_{\mathrm{A}}Q(0)_{\mathrm{R}}}|0\rangle \\ &= \lim_{L\to\infty}\exp\Big\{-\frac{\alpha_{\mathrm{A}}L}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}k^1 \\ &\quad\times [a(k^1) - a^{\dagger}(k^1)]\mathrm{e}^{-L^2(k^1)^2/4}\Big\}|0\rangle.\end{aligned} \tag{8.12}$$

The energy operator of the massless scalar field described by the Lagrangian (8.1) is equal to

$$\begin{aligned}\hat{H}(t) &= \int_{-\infty}^{\infty}\mathrm{d}x\mathcal{H}(x,t) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x : \Big[\Big(\frac{\partial\varphi(x,t)}{\partial t}\Big)^2 + \Big(\frac{\partial\varphi(x,t)}{\partial x}\Big)^2\Big] : \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k^1}{2\pi}a^{\dagger}(k^1)a(k^1).\end{aligned} \tag{8.13}$$

One can easily show that the wave functions $|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}}$ are eigenfunctions of the energy operator (8.13) for the eigenvalue zero

$$\hat{H}(t)|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}} = E(\alpha_{\mathrm{A}})|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}} = 0. \tag{8.14}$$

This is evidence that the energy of the vacuum state is infinitely degenerated, and the vacuum state is not invariant under the field translations (8.3). The wave functions of the vacuum state $|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}}$ and $|\alpha'_{\mathrm{A}}\rangle_{\mathrm{R}}$ are not orthogonal to each other for $\alpha'_{\mathrm{A}} \neq \alpha_{\mathrm{A}}$ and the scalar product ${}_{\mathrm{R}}\langle\alpha'_{\mathrm{A}}|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}}$ amounts to [40]

$${}_{\mathrm{R}}\langle\alpha'_{\mathrm{A}}|\alpha_{\mathrm{A}}\rangle_{\mathrm{R}} = \mathrm{e}^{-(\alpha'_{\mathrm{A}}-\alpha_{\mathrm{A}})^2/2}. \tag{8.15}$$

However, since the eigenvalue of the wave functions $|\alpha_{\mathrm{A}}\rangle$ is zero, they can be orthogonalized by any appropriate orthogonalization procedure as used in molecular and nuclear physics.

We would like to emphasize that the results expounded above are not related to the impossibility to determine the two-point Wightman function (8.6) in the infrared region of φ field fluctuations. In fact, the analysis of the non-invariance of the vacuum wave function under the symmetry transformations (8.3) treats the massless scalar field at the tree level. This is an appropriate description, since the massless scalar field φ is free, no vacuum fluctuations are entangled and the quanta of the massless φ field are on mass shell.

Thus, following the Itzykson–Zuber analysis of the $1+1$-dimensional massless scalar field theory of the φ field described by the Lagrangian (8.1) we have shown that

(i) the translation symmetry (8.3) is spontaneously broken,
(ii) Goldstone bosons appear and they are quanta of the φ field,
(iii) the ground state is not invariant under the field-translation symmetry and
(iv) (iv) the energy of the ground state is infinitely degenerated. Hence, all requirements for a continuous symmetry to be spontaneously broken are available in the $1 + 1$-dimensional quantum field theory of a massless scalar field described by the Lagrangian (8.1).

Now let us show that chiral symmetry in our approach to the massless Thirring model is spontaneously broken, i.e. the wave function of the ground state is not invariant under chiral rotations and a Goldstone boson exists. The fact of the non-invariance of the ground state of the massless Thirring model under chiral rotations has been demonstrated in (6.40). The Goldstone bosons are the quanta the ϑ field and the effective Lagrangian of the fermion system, quantized around the minimum of the effective potential (3.21), is invariant under chiral rotations. In order to show this we suggest to rewrite the partition function (3.29) as follows:

$$Z_{\mathrm{Th}} = \int\mathcal{D}\vartheta\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\mathrm{i}\int\mathrm{d}^2x\mathcal{L}_{\mathrm{eff}}[\bar{\psi},\psi;\vartheta], \tag{8.16}$$

where the effective Lagrangian $\mathcal{L}_{\rm eff}[\bar{\psi},\psi;\vartheta]$ is determined by

$$\mathcal{L}_{\rm eff}[\bar{\psi},\psi;\vartheta] = \bar{\psi}(x)(\mathrm{i}\gamma^{\mu}\partial_{\mu} - M\mathrm{e}^{\mathrm{i}\gamma^5\beta\vartheta(x)})\psi(x). \quad (8.17)$$

For convenience we have renormalized the ϑ field and the parameter β is given by (4.5). The Lagrangian (8.17) is invariant under chiral rotations (6.38)

$$\begin{aligned}\mathcal{L}_{\rm eff}[\bar{\psi},\psi;\vartheta] &\to \mathcal{L}_{\rm eff}[\bar{\psi}',\psi';\vartheta'] \\ &= \bar{\psi}'(x)(\mathrm{i}\gamma^{\mu}\partial_{\mu} - M\mathrm{e}^{\mathrm{i}\gamma^5\beta\vartheta'(x)})\psi'(x), \quad (8.18)\end{aligned}$$

where the field $\vartheta'(x)$ is defined by

$$\vartheta'(x) = \vartheta(x) - 2\alpha_{\rm A}/\beta. \quad (8.19)$$

Such an invariance under chiral rotations is trivially explained by the Mexican-hat shape of the effective potential (3.21) depicted in Fig. 2 allowing arbitrary changes of the ϑ field around the hill at fixed ρ.

In the bosonic description the effective Lagrangian (8.17) is equivalent to the effective Lagrangian depending only on the ϑ field (3.35)

$$\mathcal{L}_{\rm eff}[\bar{\psi},\psi;\vartheta] \to \mathcal{L}_{\rm eff}[\vartheta] = \frac{1}{2}\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x), \quad (8.20)$$

which is invariant under field translations (8.19) caused by chiral rotations.

Thus, the fermionic system described by the massless Thirring model and quantized around the minimum of the effective potential (3.21) satisfies all requirements for a spontaneously broken chiral symmetry as has been discussed above (8.7)–(8.15). This is evidence that the bosonization of this fermionic system runs via the chirally broken phase and is finally described by the Goldstone boson field ϑ.

For the calculation of the effective Lagrangian (8.20) we break chiral symmetry explicitly by a local chiral rotation

$$\psi(x) \to \mathrm{e}^{-\mathrm{i}\gamma^5\beta\vartheta(x)/2}\psi(x). \quad (8.21)$$

This reduces the effective Lagrangian (8.17) to the form

$$\begin{aligned}&\mathcal{L}_{\rm eff}[\bar{\psi},\psi;\vartheta] \\ &= \bar{\psi}(x)\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} + \frac{1}{2}\beta\gamma^{\mu}\varepsilon_{\mu\nu}\partial^{\nu}\vartheta(x) - M\right)\psi(x). \quad (8.22)\end{aligned}$$

The term proportional to M breaks chiral symmetry explicitly

$$\begin{aligned}&\mathcal{L}_{\rm eff}[\bar{\psi},\psi;\vartheta] \to \mathcal{L}_{\rm eff}[\bar{\psi}',\psi';\vartheta'] \quad (8.23)\\ &= \bar{\psi}'(x)\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} + \frac{1}{2}\beta\gamma^{\mu}\varepsilon_{\mu\nu}\partial^{\nu}\vartheta(x) - M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}}\right)\psi'(x).\end{aligned}$$

Such a violation of chiral invariance is caused by a gauge fixing specifying the starting point at the Mexican-hat for counting of the chiral phase of a fermion field during a travel around the hill. Due to the Abelian symmetry Faddeev–Popov ghosts do not appear and the Faddeev–Popov determinant is equal to unity.

Let us show that such a gauge fixing does not affect the result. The evaluation of the effective Lagrangian of the ϑ field does not depend on $\alpha_{\rm A}$ and coincides with (8.20).

Indeed, the effective Lagrangian of the ϑ field is defined by the two-vertex fermion diagram

$$\begin{aligned}\mathcal{L}_{\rm eff}(x) &= -\frac{1}{8\pi}\int\frac{\mathrm{d}^2x_1\mathrm{d}^2k_1}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}k_1\cdot(x_1-x)}A_{\mu}(x)A_{\nu}(x_1)\\ &\quad\times\int\frac{\mathrm{d}^2k}{\pi\mathrm{i}}\mathrm{tr}\left\{\frac{1}{M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}} - \hat{k}}\gamma^{\mu}\right.\\ &\quad\left.\times\frac{1}{M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}} - \hat{k} - \hat{k}_1}\gamma^{\nu}\right\}\\ &= -\frac{1}{8\pi}\int\frac{\mathrm{d}^2x_1\mathrm{d}^2k_1}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}k_1\cdot(x_1-x)}A_{\mu}(x)A_{\nu}(x_1)\\ &\quad\times\int\frac{\mathrm{d}^2k}{\pi\mathrm{i}}\mathrm{tr}\left\{\frac{M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}} + \hat{k}}{M^2 - k^2}\gamma^{\mu}\right.\\ &\quad\left.\times\frac{M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}} + \hat{k} + \hat{k}_1}{M^2 - (k+k_1)^2}\gamma^{\nu}\right\}\\ &= -\frac{1}{8\pi}\int\frac{\mathrm{d}^2x_1\mathrm{d}^2k_1}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}k_1\cdot(x_1-x)}A_{\mu}(x)A_{\nu}(x_1)\\ &\quad\times\int\frac{\mathrm{d}^2k}{\pi\mathrm{i}}\mathrm{tr}\left\{\frac{1}{M^2-k^2}\frac{1}{M^2-(k+k_1)^2}\right.\\ &\quad\times\Big[M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}}\gamma^{\mu}M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}}\gamma^{\nu}\\ &\quad+\hat{k}\gamma^{\mu}M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_{\rm A}}\gamma^{\nu} + M\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha_A}\gamma^{\mu}(\hat{k}+\hat{k}_1)\gamma^{\nu}\\ &\quad\left.+\,\hat{k}\gamma^{\mu}(\hat{k}+\hat{k}_1)\gamma^{\nu}\Big]\right\}. \quad (8.24)\end{aligned}$$

The second and the third terms vanish due to the trace over Dirac matrices. Therefore, the r.h.s. of (8.24) does not depend on $\alpha_{\rm A}$ and the result of the evaluation of the momentum integral leads to the effective Lagrangian (8.20). This Lagrangian of the ϑ field is invariant under ϑ field translations (8.19) caused by chiral rotations. As the vacuum of the ϑ field is not invariant under (8.19), the symmetry becomes spontaneously broken in the way described above and the Goldstone boson is the quantum of the ϑ field [40].

This result agrees completely with the derivation of effective chiral Lagrangians within the ENJL model with chiral $U(N_F)\times U(N_F)$ [19–22], where N_F is the number of quark flavors. In fact, the starting Lagrangian of the ENJL model with massless quark fields is invariant under the chiral group $U(N_F)\times U(N_F)$. Then, via the chirally broken phase with dynamical quarks the Lagrangian of the ENJL model after the integration over quark degrees of freedom acquires the form of the effective chiral Lagrangian containing only local boson fields. This effective chiral Lagrangian is invariant under the chiral group $U(N_F)\times U(N_F)$ [19–22]. However, the bosonic system described by this effective chiral Lagrangian is not stable under symmetry breaking, and the phase of spontaneously broken chiral symmetry is energetically preferable. In this spontaneously broken phase all vacuum expectation values of local bosonic fields are defined by parameters of the chirally broken phase of the quark system described by the ENJL model [19–22].

Now we are able to discuss the problem of the vanishing of the vacuum expectation value of $\bar{\psi}(x)\psi(x)$ in (4.19). Using the Abelian bosonization rules (4.17) we get

$$\begin{aligned}\langle 0|\bar{\psi}(x)\psi(x)|0\rangle &= \langle 0|[\sigma_+(x)+\sigma_-(x)]|0\rangle \\ &= \frac{1}{2}Z^{-1}\langle\bar{\psi}\psi\rangle\langle 0|[A_-(x)+A_+(x)]|0\rangle \\ &= \frac{1}{2}Z^{-1}\langle\bar{\psi}\psi\rangle \\ &\quad\times\int\mathcal{D}\vartheta\mathrm{e}^{-\mathrm{i}(1/2)\int\mathrm{d}^2z\vartheta(z)(\Box+\mu^2)\vartheta(z)} \\ &\quad\times(\mathrm{e}^{\mathrm{i}\beta\vartheta(x)}+\mathrm{e}^{-\mathrm{i}\beta\vartheta(x)}) \\ &= Z^{-1}\langle\bar{\psi}\psi\rangle\mathrm{e}^{(1/2)\beta^2\mathrm{i}\Delta(0)} \\ &= Z^{-1}\langle\bar{\psi}\psi\rangle\left(\frac{\mu}{M}\right)^{\beta^2/4\pi}. \end{aligned}\tag{8.25}$$

The cut-off μ has been introduced by Coleman [3] in order to regularized the two-point Green function of the ϑ field in the infrared region. Therefore, a regularized correlation function should be obtained in the limit $\mu\to 0$. Setting $\mu=0$ we get

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\mathrm{R}} = 0. \tag{8.26}$$

We would like to emphasize that in our approach the vanishing of the fermion condensate (8.26) is not due to the triviality of the vacuum. Our vacuum is essentially different from the vacua which were used in [3, 5–16] as we have explained in Sects. 4–6.

As has been stated by Itzykson and Zuber [27] such a vanishing of the correlation function $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle$ is caused by the sorrowful fact that a massless scalar field theory is *undefined* in a 1 + 1-dimensional world due to severe infrared divergences. [27]. The former leads to a randomization of the ϑ field in the infrared region that averages $\langle 0|[A_-(x)+A_+(x)]|0\rangle$ to zero [27, 28].

Since the problem of the vanishing of the correlation function (8.26) is related to full extent to the definition of the massless scalar field in 1 + 1-dimensional space-time, we suggest to regularize the correlation function (8.25) within dimensional regularization. In dimensional regularization the two-point Green function of the ϑ field described by the effective Lagrangian (8.20) is defined by

$$\begin{aligned}\mathrm{i}\Delta(x-y) &= \int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\frac{1}{p^2+\mathrm{i}0}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \\ &\to\int\frac{\mathrm{d}^dp}{(2\pi)^d\mathrm{i}}\frac{\lambda^{2-d}}{p^2+\mathrm{i}0}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \\ &= -\frac{1}{4\pi^{d/2}}[-\lambda^2(x-y)^2]^{((2-d)/2)}\Gamma\left(\frac{d-2}{2}\right),\end{aligned}\tag{8.27}$$

where λ is a dimensional parameter making the Green function dimensionless. We fix this parameter below. In order to obtain the regularized value $\mathrm{i}\Delta(0)_{\mathrm{R}}$, we keep $d=2-\varepsilon$ and set $x-y=0$. This yields $\mathrm{i}\Delta(0)_{\mathrm{R}}=0$.

The same result can be obtained within analytical regularization

$$\begin{aligned}\mathrm{i}\Delta(x-y) &= \int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\frac{1}{p^2+\mathrm{i}0}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \\ &\to-\int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\frac{\lambda^{2\alpha-2}}{(-p^2+\mathrm{i}0)^{\alpha}}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \\ &= -\frac{1}{4^{\alpha}\pi}[-\lambda^2(x-y)^2]^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}.\end{aligned}\tag{8.28}$$

Keeping $\alpha=1+\varepsilon/2$ at $\varepsilon\to+0$ we get again $\mathrm{i}\Delta(0)_{\mathrm{R}}=0$.

Using the regularized Green function $\mathrm{i}\Delta(0)_{\mathrm{R}}=0$ the vacuum expectation value of the fermion condensate (8.25) is equal to

$$\begin{aligned}\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\mathrm{R}} &= Z^{-1}\langle\bar{\psi}\psi\rangle\mathrm{e}^{(1/2)\beta^2\mathrm{i}\Delta(0)_{\mathrm{R}}} \\ &= Z^{-1}\langle\bar{\psi}\psi\rangle.\end{aligned}\tag{8.29}$$

Since there are no divergences we should set $Z=1$. This gives the fermion condensate

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\mathrm{R}} = \langle\bar{\psi}\psi\rangle, \tag{8.30}$$

which is in complete agreement with our result obtain within the BCS formalism.

Thus, we have shown that the problem of the fermion condensate, averaged to zero by the ϑ field fluctuations, can be avoided by using dimensional or analytical regularization.

The solution of the massless Thirring model in the sense of an explicit evaluation of any correlation function

$$\left\langle 0\left|T\left(\prod_{i=1}^{p}\prod_{j=1}^{n}\sigma_+(x_i)\sigma_-(y_j)\right)\right|0\right\rangle \tag{8.31}$$

runs as follows. Using the Abelian bosonization rules (4.17) the fermion correlation function (8.31) reduces to the ϑ field correlation function

$$\begin{aligned}&\left\langle 0\left|T\left(\prod_{i=1}^{p}\prod_{j=1}^{n}\sigma_+(x_i)\sigma_-(y_j)\right)\right|0\right\rangle \\ &= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{(2Z)^{p+n}}\left\langle 0\left|T\left(\prod_{i=1}^{p}\prod_{j=1}^{n}A_-(x_i)A_+(y_j)\right)\right|0\right\rangle \\ &= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{(2Z)^{p+n}} \\ &\times\int\mathcal{D}\vartheta\mathrm{e}^{-\mathrm{i}(1/2)\int\mathrm{d}^2x\vartheta(x)\Box\vartheta(x)}A_-(x_i)A_+(y_j) \\ &= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{(2Z)^{p+n}}\exp\left\{\frac{1}{2}\beta^2(p+n)\mathrm{i}\Delta(0)\right\} \\ &\times\exp\left\{\beta^2\sum_{j<k}^{p}\mathrm{i}\Delta(x_j-x_k)\right.\end{aligned}\tag{8.32}$$

$$+ \beta^2 \sum_{j<k}^{n} \mathrm{i}\Delta(y_j - y_k) - \beta^2 \sum_{j=1}^{p}\sum_{k=1}^{n} \mathrm{i}\Delta(x_j - y_k)\Bigg\}$$

$$= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{(2Z)^{p+n}} \exp\Bigg\{\beta^2 \sum_{j<k}^{p} \mathrm{i}\Delta(x_j - x_k)$$

$$+ \beta^2 \sum_{j<k}^{n} \mathrm{i}\Delta(y_j - y_k) - \beta^2 \sum_{j=1}^{p}\sum_{k=1}^{n} \mathrm{i}\Delta(x_j - y_k)\Bigg\},$$

where we have used $\mathrm{i}\Delta(0)_{\mathrm{R}} = 0$ obtained within dimensional and analytical regularization. Assuming now that all relative distances do not vanish, we should take the limits $\varepsilon \to +0$ and take away the dimensional or analytical regularization of the two-point Green functions. In the limit $\varepsilon \to +0$ the Green function $\mathrm{i}\Delta(x-y)$ reads

$$\mathrm{i}\Delta(x-y) = \frac{1}{2\pi\varepsilon} + \frac{1}{4\pi}\ln[-\lambda^2(x-y)^2]. \tag{8.33}$$

Absorbing the divergent parts of the Green functions in the constant Z^{p+n} we obtain the regularized correlation function

$$\left\langle 0\left|T\left(\prod_{i=1}^{p}\prod_{j=1}^{n}\sigma_+(x_i)\sigma_-(y_j)\right)\right|0\right\rangle \tag{8.34}$$

$$= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{2^{p+n}}$$

$$\times\frac{\displaystyle\prod_{j<k}^{p}[-\lambda^2(x_j-x_k)^2]^{\beta^2/4\pi}\prod_{j<k}^{n}[-\lambda^2(y_j-y_k)^2]^{\beta^2/4\pi}}{\displaystyle\prod_{j=1}^{p}\prod_{k=1}^{n}[-\lambda^2(x_j-y_k)^2]^{\beta^2/4\pi}}$$

$$= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{2^{p+n}}\lambda^{(\beta^2/(4\pi))[(p-n)^2-(p+n)]}$$

$$\times\frac{\displaystyle\prod_{j<k}^{p}[-(x_j-x_k)^2]^{\beta^2/4\pi}\prod_{j<k}^{n}[-(y_j-y_k)^2]^{\beta^2/4\pi}}{\displaystyle\prod_{j=1}^{p}\prod_{k=1}^{n}[-(x_j-y_k)^2]^{\beta^2/4\pi}}.$$

In order to fix a parameter λ we suggest to compare our expression for the correlation function of self-coupled fermion fields (8.34) with the correlation function of free fermion fields calculated by Klaiber [5] for $p = n$ and space-like distances:

$$\left\langle 0\left|T\left(\prod_{i=1}^{n}\sigma_+(x_i)\sigma_-(y_i)\right)\right|0\right\rangle \tag{8.35}$$

$$= \frac{1}{(2\pi)^{2n}}\frac{\displaystyle\prod_{j<k}^{n}[-(x_j-x_k)^2]\prod_{j<k}^{n}[-(y_j-y_k)^2]}{\displaystyle\prod_{j=1}^{n}\prod_{k=1}^{n}[-(x_j-y_k)^2]}.$$

Taking the *mathematical* limit $\beta^2/4\pi \to 1$ and setting $p = n$ we obtain from the comparison of (8.34) and (8.35) that $\lambda = \pm\pi\langle\bar{\psi}\psi\rangle$. It is reasonable to choose λ positive, $\lambda = -\pi\langle\bar{\psi}\psi\rangle$. Using this expression for λ we recast the correlation function (8.34) into the form

$$\left\langle 0\left|T(\prod_{i=1}^{p}\prod_{j=1}^{n}\sigma_+(x_i)\sigma_-(y_j))\right|0\right\rangle \tag{8.36}$$

$$= \frac{\langle\bar{\psi}\psi\rangle^{p+n}}{2^{p+n}}[-\pi\langle\bar{\psi}\psi\rangle]^{(\beta^2/(4\pi))[(p-n)^2-(p+n)]}$$

$$\times\frac{\displaystyle\prod_{j<k}^{p}[-(x_j-x_k)^2]^{\beta^2/4\pi}\prod_{j<k}^{n}[-(y_j-y_k)^2]^{\beta^2/4\pi}}{\displaystyle\prod_{j=1}^{p}\prod_{k=1}^{n}[-(x_j-y_k)^2]^{\beta^2/4\pi}}.$$

Thus, dimensional (or analytical) regularization of the theory for the massless scalar ϑ field leads to the expressions for correlation functions which agree fully with the BCS formalism. We should emphasize that the r.h.s. of (8.34) does not vanish even if $p \neq n$. Recall, that the vacuum expectation value (8.30) calculated for the trivial chiral invariant vacuum vanishes for $p \neq n$ [3,5]. The explicit evaluation of the correlation function (8.31) in the form (8.36) implies the solution of the massless Thirring model in our approach.

The obtained results show that a massless scalar field theory in 1 + 1-dimensional space-time is ill-defined in agreement with Coleman's statements. Therefore, in the infrared region there are no single-particle Goldstone states [26]. The Goldstone bosons being the quanta of the ϑ field exist in the infrared region in the form of randomized ensemble. The fermion condensate, averaged over the ϑ field, vanishes due to the contribution of the randomized ensemble of infrared Goldstone bosons. Since it is fully a dimensional problem the application of dimensional (or analytical) regularization allows one to escape the problem of the randomization of low-frequency quanta of the massless scalar field ϑ and calculate the non-vanishing value of the fermion condensate averaged over the ϑ field fluctuations in agreement with the result obtained within the BCS formalism (6.32) and (1.14).

9 Conclusion

We investigated the problem of the solution of the massless Thirring model and the equivalence between the massive Thirring model and the SG model in the chirally broken phase of the fermion system. We found that the fermion system described by the massless Thirring model, invariant under the chiral group $U_{\mathrm{V}}(1) \times U_{\mathrm{A}}(1)$, possesses a chiral symmetric phase with a trivial perturbative vacuum and a phase of spontaneously broken chiral symmetry with a non-perturbative vacuum. We have shown that the ground state of the massless Thirring model in the chirally broken phase coincides with the ground of the

BCS theory of superconductivity. Using the wave function of the ground state in the BCS theory of superconductivity we have calculated the energy density of the non-perturbative vacuum $\mathcal{E}(M)$ in the massless Thirring model. We have shown that the energy density of the non-perturbative vacuum $\mathcal{E}(M)$ coincides with the effective potential $V[M]$ (3.21), which is defined by the integration over fermion field fluctuations, and acquires a minimum when the dynamical mass M of fermions satisfies the gap equation (1.14). The mass spectrum of vacuum fluctuations of fermions is restricted from below. In fact, when M is kept constant at $\Lambda \to \infty$ the energy density tends to the limit $\mathcal{E}(M) = V[M] \to -M^2/4\pi$.

The chiral symmetric phase corresponds to a system with massless fermions, $M = 0$, and vanishing fermion condensate. The chirally broken phase is characterized by (i) a non-zero value of a fermion condensate, (ii) the appearance of dynamical fermions with a dynamical mass $M \neq 0$ and (iii) fermion–anti-fermion pairing [18]. The energy density of the ground state of the fermion system $\mathcal{E}(M)$ reaches a maximum, $\mathcal{E}(0) = 0$, in the chiral symmetric phase and it is negative, $\mathcal{E}(M) < 0$, in the chirally broken one. Hence, the chirally broken phase is energetically preferable and the Thirring model should be bosonized in the chirally broken phase accompanied by fermion–anti-fermion pairing.

Using the path integral technique we bosonized explicitly the massless Thirring model. We have shown that in the bosonic description the massless Thirring model is a quantum field theory of a massless free scalar field $\vartheta(x)$. The generating functional of Green functions in the massless Thirring model can be expressed in terms of a path integral over the massless scalar field $\vartheta(x)$ coupled to external sources of fermion fields via $A_\pm(x) = \mathrm{e}^{\pm \mathrm{i}\beta\vartheta(x)}$ couplings. This allows one to represent any Green function in the massless Thirring model in the fermionic description by a Gaussian path integral of products of $A_\pm(x) = \mathrm{e}^{\pm \mathrm{i}\beta\vartheta(x)}$ couplings in the bosonic formulation. Since these Gaussian path integrals can be calculated explicitly, this provides a solution of the massless Thirring model.

The evaluation of correlation functions of massless Thirring fermions by means of the integration over massless ϑ field fluctuations is related to the Mermin–Wagner theorem [25], Hohenberg's [37] and Coleman's [26] proofs concerning the vanishing of long-range order for systems with a continuous symmetry described by quantum field theories in two-dimensional space [25,38] and $1 + 1$-dimensional space-time [26]. The vanishing of the long-range order parameter implies that there is no spontaneously broken continuous symmetry in quantum field theories defined in two-dimensional space and a $1 + 1$-dimensional space-time.

Coleman's proof of this statement has been focused upon the impossibility to define a free massless scalar field theory in a $1 + 1$-dimensional space-time. Coleman found that a free massless scalar field theory is ill-defined due to meaningless infrared divergences that screen fully one-particle Goldstone boson states. This screening of a pole singularity in the Fourier transform of the two-point Wightman function is formulated by Coleman as the absence of the Goldstone bosons in a free massless scalar field theory in a $1 + 1$-dimensional space-time. This statement has been extended by Coleman onto any quantum system with a continuous symmetry embedded in a $1 + 1$-dimensional space-time [26].

The relation of Coleman's statement to the Mermin–Wagner–Hohenberg theorem [25,38] runs in the way explained, for example, in [27,28]. In fact, the infrared divergences of a free massless scalar fields lead to the appearance of a randomized ensemble of very low-frequency quanta of a massless scalar field. Due to this randomization the fermion condensate, proportional to $\cos\beta\vartheta(x)$, averages over the ϑ field fluctuations to zero.

Using Itzykson–Zuber's analysis of a free massless scalar field theory [40] and adjusting it to 1+1-dimensional space-time we have shown that (i) a continuous symmetry related to global scalar field translations is spontaneously broken, (ii) the vacuum wave function is not invariant under symmetry transformations and the vacuum energy level is infinitely degenerated. Following Itzykson and Zuber [40] we argue that Goldstone bosons appear as quanta of a free massless scalar field.

Accepting this point that a continuous symmetry of a quantum field theory of a free massless scalar field can be spontaneously broken, we have suggested that the problem of the vanishing of the fermion condensate in the massless Thirring model, averaged over the randomized ensemble of low-frequency quanta of the θ field, can be solved within an appropriate regularization. We have applied dimensional and analytical regularizations. By virtue of these regularization procedures we have succeeded in smoothing the infrared behavior of the ϑ field and get the fermion condensate averaged over the ϑ field fluctuations to a non-zero value, in complete agreement with our results obtained within the Nambu–Jona–Lasinio prescription (1.13)–(1.16) and the BCS formalism (6.32).

The bosonization of the massive Thirring model runs parallel the bosonization of the massless one. Starting with the fermion system in the phase of spontaneously broken chiral symmetry we arrive at the bosonized version described by the SG model. The parameters of the SG model can be expressed in terms of the parameters of the massive Thirring model and read

$$\alpha = -\beta^2 m\langle\bar{\psi}\psi\rangle + \frac{m^2}{g}\beta^2,$$

where the fermion condensate is defined by (1.16) and the coupling constant β depends on g via relation (4.5)

$$\frac{8\pi}{\beta^2} = 1 - \mathrm{e}^{-2\pi/g}.$$

The new relation between β and g leads to the fact that in our approach the coupling constant β^2 is always greater than 8π, $\beta^2 > 8\pi$. This disagreement with Coleman [3] is caused by different initial conditions for the evolution of the fermion system described by the Thirring model. In fact, when the fermion system evolves in the chiral symmetric phase Coleman's relation between β and g is valid. In turn, if the fermion system starts with the chirally broken phase the bosonized version of the fermion system is described by the SG model with the relation between β and g given by (4.5) and $\alpha = -\beta^2 m\langle\bar{\psi}\psi\rangle + m^2\beta^2/g$.

The evaluation of correlation functions in the massive Thirring model in terms of the path integral over the SG model field can be carried by the Abelian bosonization rules given by (5.21)

$$Zm\bar{\psi}(x)\left(\frac{1\mp\gamma^5}{2}\right)\psi(x) = -\frac{\alpha}{2\beta^2}\mathrm{e}^{\pm \mathrm{i}\beta\vartheta(x)} + \frac{m^2}{2g}.$$

At leading order in the m expansion this expression reduces to the Abelian bosonization rules derived by Coleman (1.10) [3].

The existence of the chirally broken phase in the massless Thirring model we have also confirmed within the standard operator formalism. We have shown that starting with the chiral invariant Lagrangian and normal ordering the fermion operators in the interaction term at the scale M we arrive at the Lagrangian of the massive Thirring model for fermions with mass M only if the gap equation (1.13) is fulfilled. Using the equations of motion for the fermion fields in the massless Thirring model we have shown that the chirally broken phase is stable during the evolution of the fermion system when it started to evolve from the chirally broken phase.

The stability of the chirally broken phase with the non-perturbative vacuum could in principle be destroyed by the contribution of the fluctuations of the ρ field around the minimum of the effective potential (3.21), the $\tilde{\rho}$ field fluctuations[8]. In Appendix B we have shown that the $\tilde{\rho}$ field, rescaled in an appropriate way in order to get the correct kinetic term, acquires a mass proportional the cut-off Λ and in the limit $\Lambda \to \infty$ becomes fully decoupled from the system. This result agrees completely with the Appelquist–Carazzone decoupling theorem [41]. This testifies that the chirally broken phase with the non-perturbative vacuum cannot be ruined by fluctuations around the minimum of the effective potential and is fully determined by the effective potential (3.21).

We have revealed that the existence of the chirally broken phase in the massless Thirring model changes crucially the Schwinger term in the equal-time commutation relation $[j_0(x,t), j_1(y,t)]$. We have shown that the Schwinger term calculated for the non-perturbative vacuum in the chirally broken phase depends explicitly on the coupling constant g. For the chiral symmetric phase and the trivial vacuum the Schwinger term is equal to the value calculated previously by Sommerfield [34]. In the limit $g \to 0$ our value of the Schwinger term reduces to that obtained by Sommerfield.

Now let us clarify the physical meaning of the inequality $\beta^2 > 8\pi$ obtained in our approach. For this aim we suggest to rescale the ϑ field, $\beta\vartheta(x) \to \vartheta(x)$. Then in natural units $\hbar = c = 1$ the action S reads

$$S = \frac{1}{\beta^2}\int \mathrm{d}^2x\left[\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \alpha(\cos\vartheta(x) - 1)\right], \tag{9.1}$$

with $0 \le \vartheta(x) \le 2\pi$. This allows one to interpret β^2 in the sense of "$\hbar$" distinguishing "quantum" and "classical" states of the SG model. In the classical limit, $\beta^2 \to 0$, we arrive at a system of classical Klein–Gordon waves and solitons. The action S

$$S = \frac{1}{\beta^2}\int \mathrm{d}^2x\left[\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \beta^2\left(-m\langle\bar{\psi}\psi\rangle + \frac{m^2}{g}\right)(\cos\vartheta(x) - 1)\right], \tag{9.2}$$

where for simplicity we have kept the leading terms in the m expansion, describes in the $\beta^2 \to 0$ limit a theory of massless classical ϑ-waves

$$\vartheta(x,t) = \vartheta_-(t-x) + \vartheta_+(t+x) \tag{9.3}$$

obeying

$$\Box\vartheta(x,t) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\vartheta(x,t) = 0, \tag{9.4}$$

with arbitrary functions $\vartheta_-(t-x)$ and $\vartheta_+(t+x)$. The mass of Goldstone bosons caused completely by quantum effects, $M_\vartheta = (-m\beta^2\langle\bar{\psi}\psi\rangle)^{1/2}$, vanishes in the limit $\beta^2 \to 0$. In turn, the soliton mass tends to infinity, $M_{\mathrm{sol}} \propto 1/\beta \to \infty$, and solitons decouple from the system.

For $\beta^2 > 8$ the mass of the Goldstone boson becomes greater than the mass of a single soliton:

$$\frac{M_{\mathrm{sol}}}{M_\vartheta} = \frac{8}{\beta^2} < 1. \tag{9.5}$$

This implies that in the "quantum limit", $\beta^2 \gg 1$, the creation of non-perturbative soliton configurations is energetically preferable with respect to the creation of Goldstone bosons. This yields that at $\beta^2 > 8\pi$, when $M_\vartheta \gg M_{\mathrm{sol}}$, the Goldstone bosons are decoupled from the system and there exist practically only solitons. Hence, the inequality $\beta^2 > 8\pi$ corresponds to the non-perturbative phase of the SG model populated by soliton states only.

We have shown that the topological current of the SG model coincides with the Noether current of the massive Thirring model related to the $U_{\mathrm{V}}(1)$ invariance. Since this Noether current is responsible for the conservation of the fermion number in the massive Thirring model, the topological charge of the SG model has the meaning of the fermion number. Since many-soliton solutions obey Pauli's exclusion principle, this should prove Skyrme's statement [4] that the SG model solitons can be interpreted as massive fermions. Thus, via spontaneously broken chiral symmetry the massive Thirring fermions get converted into extended particles with the properties of fermions and masses much heavier than their initial mass

$$M^2_{\mathrm{sol}} = -\frac{64}{\beta^2}m\langle\bar{\psi}\psi\rangle + O(m^2) \gg m^2.$$

[8] This question has been raised by Valerii Rubakov

Finally, we would like to mention that recently [42] one of the authors suggested a generalization of the sine-Gordon model to 3+1-dimensions. This model has also stable solitonic excitations characterized by a winding number defining a chirality for fermions. The results obtained in the present paper can be of use for the derivation of the model suggested in [42] as a bosonized version of the 3+1-dimensional NJL model with chiral $SU(2)\times SU(2)$.

Numerous applications to hadron physics of the chiral soliton model based on the linear σ model of Gell–Mann and Levy, the extended linear σ model and the Nambu–Jona–Lasinio quark model with $SU(2)\times SU(2)$ and $SU(3)\times SU(3)$ chiral symmetry one can find in papers written by Göke with co-workers starting in 1985 [43].

Acknowledgements. The authors are greatful to Walter Thirring, Wolfgang Kummer and Valerii Rubakov for fruitful discussions. We thank Jan Thomassen for critical comments on our approach at the initial stage of the work. Numerous and fruitful discussions with Max Meinhardt are greatly appreciated. This work was supported in part by Fonds zur Förderung der Wissenschaftlichen Forschung P13997-TPH[9].

Appendix A. Chiral Jacobian

In this appendix we adduce the calculation of the Jacobian induced by chiral rotations in (3.30). We show that by using an appropriate regularization scheme this Jacobian can be found to be equal to unity. We follow the procedure formulated in [12–17]. For the calculation of the chiral Jacobian we start with the Lagrangian implicitly defined in (3.29) and (3.30)

$$\begin{aligned}\mathcal{L}_{\psi}(x) &= \bar{\psi}(x)(\mathrm{i}\gamma^{\mu}\partial_{\mu} - M\mathrm{e}^{\mathrm{i}\gamma^5\vartheta(x)})\psi(x)\\ &= \bar{\psi}(x)D(x;0)\psi(x), \qquad (\mathrm{A.1})\end{aligned}$$

where $D(x;0)$ is the Dirac operator given by

$$D(x;0) = \mathrm{i}\gamma^{\mu}\partial_{\mu} - M\mathrm{e}^{\mathrm{i}\gamma^5\vartheta(x)}. \qquad (\mathrm{A.2})$$

By a chiral rotation

$$\begin{aligned}\psi(x) &= \mathrm{e}^{-\mathrm{i}\alpha\gamma^5\vartheta(x)/2}\chi(x),\\ \bar{\psi}(x) &= \bar{\chi}(x)\mathrm{e}^{-\mathrm{i}\alpha\gamma^5\vartheta(x)/2}, \qquad (\mathrm{A.3})\end{aligned}$$

where $0\le\alpha\le 1$, we reduce the Lagrangian (A.1) to the form

$$\mathcal{L}_{\chi}(x) = \bar{\chi}(x)D(x,\alpha)\chi(x). \qquad (\mathrm{A.4})$$

The Dirac operator $D(x;\alpha)$ reads

$$\begin{aligned}D(x;\alpha) = \mathrm{i}\gamma^{\mu}\partial_{\mu} &+ \frac{1}{2}\alpha\gamma^{\mu}\gamma^5\partial_{\mu}\vartheta(x)\\ &-M\mathrm{e}^{\mathrm{i}(1-\alpha)\gamma^5\vartheta(x)}. \qquad (\mathrm{A.5})\end{aligned}$$

At $\alpha = 1$ we obtain the Lagrangian

$$\begin{aligned}\mathcal{L}_{\chi}(x) &= \bar{\chi}(x)D(x,1)\chi(x) \qquad (\mathrm{A.6})\\ &= \bar{\chi}(x)\left(\mathrm{i}\gamma^{\mu}\partial_{\mu} + \frac{1}{2}\gamma^{\mu}\gamma^5\partial_{\mu}\vartheta(x) - M\right)\chi(x),\end{aligned}$$

where the term $-M\bar{\chi}(x)\chi(x)$ has the meaning of a mass term of the $\chi(x)$ field (see (3.30)).

Due to the chiral rotation (3.30) the fermionic measure changes as follows:

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = J[\vartheta]\mathcal{D}\chi\mathcal{D}\bar{\chi}. \qquad (\mathrm{A.7})$$

For the calculation of $J[\vartheta]$ we follow Fujikawa's procedure [12–17] and introduce eigenfunctions $\varphi_n(x;\alpha)$ and eigenvalues $\lambda_n(\alpha)$ of the Dirac operator $D(x;\alpha)$:

$$D(x;\alpha)\varphi_n(x;\alpha) = \lambda_n(\alpha)\varphi_n(x;\alpha). \qquad (\mathrm{A.8})$$

In terms of the eigenfunctions and eigenvalues of the Dirac operator the Jacobian $J[\vartheta]$ is defined by [13,14]

$$J[\vartheta] = \exp 2\mathrm{i}\int_0^1 \mathrm{d}\alpha w[\vartheta;\alpha], \qquad (\mathrm{A.9})$$

where the functional $w[\vartheta;\alpha]$ is given by [13,14]

$$\begin{aligned}w[\vartheta;\alpha] &= \lim_{\Lambda_{\mathrm{F}}\to\infty}\sum_n \varphi^{\dagger}_n(x;\alpha)\frac{1}{2}\gamma^5\vartheta(x)\mathrm{e}^{\mathrm{i}\lambda^2_n/\Lambda^2_{\mathrm{F}}}\varphi_n(x;\alpha)\\ &= \lim_{\Lambda_{\mathrm{F}}\to\infty}\frac{1}{2}\int\mathrm{d}^2x\vartheta(x)\\ &\quad\times\int\frac{\mathrm{d}^2k}{(2\pi)^2}\mathrm{tr}\{\gamma^5\langle x|\mathrm{e}^{\mathrm{i}D^2(x;\alpha)/\Lambda^2_{\mathrm{F}}}|x\rangle\}. \qquad (\mathrm{A.10})\end{aligned}$$

For the calculation of the matrix element $\langle x|\ldots|x\rangle$ we use plane waves [12–17] and get

$$\begin{aligned}&\langle x|\mathrm{e}^{\mathrm{i}D^2(x;\alpha)/\Lambda^2_{\mathrm{F}}}|x\rangle\\ &= \exp\left\{\frac{\mathrm{i}}{\Lambda^2_{\mathrm{F}}}\left[k^2 - 2M\cos((1-\alpha)\vartheta(x))\gamma^{\mu}k_{\mu}\right.\right.\\ &+\frac{1}{2}\mathrm{i}\alpha\gamma^{\mu}\gamma^{\nu}\gamma^5\partial_{\mu}\partial_{\nu}\vartheta(x) + M^2\mathrm{e}^{2\mathrm{i}(1-\alpha)\gamma^5\vartheta(x)}\\ &+(1-2\alpha)M\gamma^{\mu}\gamma^5\partial_{\mu}\vartheta(x)\cos((1-\alpha)\vartheta(x))\\ &+\mathrm{i}(1-\alpha)M\gamma^{\mu}\partial_{\mu}\vartheta(x)\sin((1-\alpha)\vartheta(x))\\ &\left.\left.-\frac{1}{4}\alpha^2\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x)\right]\right\}. \qquad (\mathrm{A.11})\end{aligned}$$

Substituting (A.11) in (A.10) we obtain

$$\begin{aligned}&\lim_{\Lambda_{\mathrm{F}}\to\infty}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\mathrm{tr}\{\gamma^5\langle x|\mathrm{e}^{\mathrm{i}D^2(x;\alpha)/\Lambda^2_{\mathrm{F}}}|x\rangle\} \qquad (\mathrm{A.12})\\ &= \frac{1}{4\pi}\mathrm{tr}\left\{-\frac{1}{2}\alpha\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu}\vartheta(x)\right.\\ &\left.- M^2\sin(2(1-\alpha)\vartheta(x))\right\}\\ &= \frac{1}{4\pi}[-\alpha\partial^{\mu}\partial_{\mu}\vartheta(x) - 2M^2\sin(2(1-\alpha)\vartheta(x))].\end{aligned}$$

[9] After this paper has been completed we became aware of the paper by Vigman and Larkin [38]. In a $1+1$-dimensional chiral invariant model with four-fermion interactions Vigman and Larkin investigated the problem of the appearance of a fermion mass. Analyzing the infrared asymptotic behavior of the one-particle Green function in the approximation of a large number of fermion fields Vigman and Larkin showed that fermions become massive as a result of four-fermion interactions. The vanishing of the fermion condensate has been declared as the absence of *spontaneous symmetry breakdown*

The functional $w[\vartheta,\alpha]$ is given then by

$$w[\vartheta,\alpha] = \frac{1}{8\pi}\int d^2x\vartheta(x)\Big[-\alpha\partial^\mu\partial_\mu\vartheta(x) -2M^2\sin(2(1-\alpha)\vartheta(x))\Big]. \tag{A.13}$$

Inserting $w[\vartheta,\alpha]$ into (A.9) and integrating over α we get the Jacobian

$$J[\vartheta] = \exp \mathrm{i}\int d^2x\Big[\frac{1}{8\pi}\partial^\mu\vartheta(x)\partial_\mu\vartheta(x) +\frac{M^2}{4\pi}(\cos 2\vartheta(x) - 1)\Big]. \tag{A.14}$$

For the derivation of the first term we have integrated by parts and dropped the surface contributions.

Our result agrees well with that obtained by Dorn for the massive Schwinger model [16]. However, Dorn pointed out that the term proportional to M^2 is *renormalization-scheme dependent* and it is *unambiguously defined if one insists on vector gauge invariance* [13,15].

Since the Thirring model is not vector gauge invariant, the term proportional to M^2 may not be well defined and may, in principle, be removed by an appropriate regularization. First, let us give physical reasons for the absence of this term. Indeed, the term in (A.14) proportional to M^2 does not depend on the derivatives $\partial_\mu\vartheta(x)$. Therefore, it contributes to the effective potential. However, we have calculated the effective potential explicitly in (3.5)–(3.21) and have shown that it does not depend on the ϑ field. Therefore, one can conclude that the determinant (3.29),

$$\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu - M e^{\mathrm{i}\gamma^5\vartheta}), \tag{A.15}$$

is invariant under global rotations, $\vartheta(x)\to\vartheta(x)+\theta$, where θ is an arbitrary constant. On the other hand, after the chiral rotation (3.30)

$$\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu - M e^{\mathrm{i}\gamma^5\vartheta}) = J[\vartheta]\mathrm{Det}\left(\mathrm{i}\gamma^\mu\partial_\mu + \frac{1}{2}\varepsilon^{\mu\nu}\gamma_\mu\partial_\nu\vartheta - M\right), \tag{A.16}$$

the determinant

$$\mathrm{Det}\left(\mathrm{i}\gamma^\mu\partial_\mu + \frac{1}{2}\varepsilon^{\mu\nu}\gamma_\mu\partial_\nu\vartheta - M\right), \tag{A.17}$$

depending only on the gradient of the ϑ field, is also invariant under global rotations, $\vartheta(x)\to\vartheta(x)+\theta$. This proves that the Jacobian $J[\vartheta]$ should not violate invariance under global rotations, $\vartheta(x)\to\vartheta(x)+\theta$, and the presence of the term proportional to M^2 in (A.14) is a problem of the regularization procedure.

In order to confirm our statement mathematically we suggest to use for the evaluation of the Jacobian the regularization procedure expounded in [44]. The functional $w[\vartheta;\alpha]$ is determined as follows:

$$\begin{aligned} w[\vartheta;\alpha] &= \sum_n \varphi^\dagger_n(x;\alpha)\frac{1}{2}\gamma^5\vartheta(x)\varphi_n(x;\alpha) \\ &= \lim_{s\to 0}\sum_n \varphi^\dagger_n(x;\alpha)\frac{1}{2}\gamma^5\vartheta(x)\lambda_n^{-s}(\alpha)\varphi_n(x;\alpha) \\ &= \lim_{s\to 0}\frac{1}{2\pi i}\oint_C d\lambda\lambda^{-s}\int d^2x\frac{1}{2}\vartheta(x) \\ &\quad\times\int\frac{d^2k}{(2\pi)^2}\varphi^\dagger_n(x;\alpha)\gamma^5\frac{1}{\lambda-\lambda_n(\alpha)}\varphi_n(x;\alpha) \\ &= \lim_{s\to 0}\frac{1}{2\pi \mathrm{i}}\oint_C d\lambda\lambda^{-s}\int d^2x\frac{1}{2}\vartheta(x) \\ &\quad\times\int\frac{d^2k}{(2\pi)^2}\varphi^\dagger_n(x;\alpha)\gamma^5\frac{1}{\lambda-D(x;\alpha)}\varphi_n(x;\alpha) \\ &= \lim_{s\to 0}\frac{1}{2\pi \mathrm{i}}\oint_C d\lambda\lambda^{-s}\int d^2x\frac{1}{2}\vartheta(x) \\ &\quad\times\int\frac{d^2k}{(2\pi)^2}\mathrm{tr}\left\{\langle x|\gamma^5\frac{1}{\lambda-D(x;\alpha)}|x\rangle\right\}. \end{aligned} \tag{A.18}$$

For plane waves [44] the matrix element is given by

$$w[\vartheta;\alpha] = \lim_{s\to 0}\frac{1}{2\pi i}\oint_C d\lambda\lambda^{-s}\int d^2x\frac{1}{2}\vartheta(x)\int\frac{d^2k}{(2\pi)^2} \times\mathrm{tr}\left\{\gamma^5\frac{1}{(\lambda-S(x;\alpha)) - \mathrm{i}\gamma^5P(x;\alpha) - (\hat{k}+\hat{A}(x;\alpha))}\right\}, \tag{A.19}$$

where we have denoted

$$\begin{aligned} \hat{A}(x;\alpha) &= \gamma^\mu\frac{1}{2}\alpha\varepsilon_{\mu\nu}\partial^\nu\vartheta(x), \\ S(x;\alpha) &= M\cos((1-\alpha)\vartheta(x)), \\ P(x;\alpha) &= M\sin((1-\alpha)\vartheta(x)). \end{aligned} \tag{A.20}$$

Due to the algebra of Dirac matrices the r.h.s. of (A.19) can be reduced to the form

$$w[\vartheta;\alpha] = \lim_{s\to 0}\frac{1}{2\pi \mathrm{i}}\oint_C d\lambda\lambda^{-s}\int d^2x\int\frac{d^2k}{(2\pi)^2} \times\frac{\mathrm{i}M\vartheta(x)\sin((1-\alpha)\vartheta(x))}{\lambda^2+M^2-2\lambda M\cos((1-\alpha)\vartheta(x)) - (k+A(x;\alpha))^2}, \tag{A.21}$$

As the integral over k is logarithmically divergent, it does not depend on a shift of k. Making a shift $k+A(x;\alpha)\to k$ we arrive at the expression

$$\begin{aligned} w[\vartheta;\alpha] &= \lim_{s\to 0}\frac{1}{2\pi \mathrm{i}}\oint_C d\lambda\lambda^{-s}\int d^2x\int\frac{d^2k}{(2\pi)^2} \\ &\quad\times\frac{\mathrm{i}M\vartheta(x)\sin((1-\alpha)\vartheta(x))}{\lambda^2+M^2-2\lambda M\cos((1-\alpha)\vartheta(x)) - k^2} \\ &= -\lim_{s\to 0}\frac{1}{2\pi \mathrm{i}}\oint_C d\lambda\lambda^{-s}\int d^2x\int\frac{d^2k_{\mathrm{E}}}{(2\pi)^2} \\ &\quad\times\frac{M\vartheta(x)\sin((1-\alpha)\vartheta(x))}{\lambda^2+M^2-2\lambda M\cos((1-\alpha)\vartheta(x)) + k^2_{\mathrm{E}}}, \end{aligned} \tag{A.22}$$

where we have passed to Euclidean momentum space.

The chiral Jacobian is now defined by

$$J[\vartheta] = \exp\left\{2\mathrm{i}\int^1_0 d\alpha w[\vartheta;\alpha]\right\} \tag{A.23}$$

$$= \exp\left\{-2\mathrm{i}\lim_{s\to 0}\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}\lambda\lambda^{-s}\int \mathrm{d}^2x\int\frac{\mathrm{d}^2k_{\mathrm{E}}}{(2\pi)^2}\right.$$
$$\left.\times\int_0^1 \mathrm{d}\alpha\frac{M\vartheta(x)\sin((1-\alpha)\vartheta(x))}{\lambda^2+M^2-2\lambda M\cos((1-\alpha)\vartheta(x))+k_{\mathrm{E}}^2}\right\}.$$

Integrating over α we obtain

$$J[\vartheta] = \exp\left\{-2\mathrm{i}\int \mathrm{d}^2x\int\frac{\mathrm{d}^2k_{\mathrm{E}}}{8\pi^2}\lim_{s\to 0}\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}\lambda\lambda^{-s-1}\right.$$
$$\left.\times\ln\left[\frac{k_{\mathrm{E}}^2+(\lambda-M)^2}{k_{\mathrm{E}}^2+(\lambda-M)^2-2\lambda M(\cos\vartheta(x)-1)}\right]\right\}. \tag{A.24}$$

Taking the limit $s\to 0$ we get

$$J[\vartheta] = \exp\left\{-2\mathrm{i}\int \mathrm{d}^2x\int\frac{\mathrm{d}^2k_{\mathrm{E}}}{8\pi^2}\frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}\lambda}{\lambda}\right. \tag{A.25}$$
$$\left.\times\ln\left[\frac{k_{\mathrm{E}}^2+(\lambda-M)^2}{k_{\mathrm{E}}^2+(\lambda-M)^2-2\lambda M(\cos\vartheta(x)-1)}\right]\right\}.$$

The integrand over λ has a pole singularity at $\lambda=0$. Since the contour C is closed around the pole singularities [44], integrating over λ we obtain

$$J[\vartheta]=1. \tag{A.26}$$

Thus, we have shown that the Jacobian of the chiral rotation transforming the functional determinant (A.15) into the functional determinant (A.17) gets the unit value within an appropriate regularization scheme.

Let us further show that the result $J[\vartheta]=1$ is retained even if we integrate first over k_{E}. Integrating over k_{E} we obtain

$$J[\vartheta] = \exp\left\{-2\mathrm{i}\int \mathrm{d}^2x\lim_{s\to 0}\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}\lambda\lambda^{-s}\right.$$
$$\times\left[\frac{M}{4\pi}\ln\left(\frac{\Lambda_{\mathrm{E}}^2}{M^2}\right)(\cos\vartheta(x)-1)-\frac{M}{4\pi}(\cos\vartheta(x)-1)\right.$$
$$\left.\left.\times\ \ln\left(1-2\frac{\lambda}{M}\cos\vartheta(x)-\frac{\lambda^2}{M^2}\right)\right]\right\}, \tag{A.27}$$

where Λ_{E} is an ultra-violet cut-off that should be taken in the limit $\Lambda_{\mathrm{E}}\to\infty$.

Setting $s=0$ we recast the Jacobian into the form

$$J[\vartheta] = \exp\left\{-2\mathrm{i}\int \mathrm{d}^2x\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}\lambda\left[\frac{M}{4\pi}\ln\left(\frac{\Lambda_{\mathrm{E}}^2}{M^2}\right)\right.\right.$$
$$\times(\cos\vartheta(x)-1)-\frac{M}{4\pi}(\cos\vartheta(x)-1)$$
$$\left.\left.\times\ \ln\left(1-2\frac{\lambda}{M}\cos\vartheta(x)+\frac{\lambda^2}{M^2}\right)\right]\right\}, \tag{A.28}$$

The first term is equal to zero due to Cauchy's theorem. Thus, the Jacobian is determined by

$$J[\vartheta] = \exp\left\{\mathrm{i}\int \mathrm{d}^2x\frac{M^2}{2\pi}(\cos\vartheta(x)-1)\right.$$
$$\left.\times\ \frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}z\ln(1-2z\cos\vartheta(x)+z^2)\right\}, \tag{A.29}$$

where we have changed variables $\lambda/M\to z$. The integral over z can be calculated as follows:

$$\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}z\ln(1-2z\cos\vartheta(x)+z^2)$$
$$=\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}z\ln[(z-\mathrm{e}^{\mathrm{i}\vartheta(x)})(z-\mathrm{e}^{-\mathrm{i}\vartheta(x)})]$$
$$=\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}z\ln(z-\mathrm{e}^{\mathrm{i}\vartheta(x)})$$
$$+\frac{1}{2\pi\mathrm{i}}\oint_C \mathrm{d}z\ln(z-\mathrm{e}^{-\mathrm{i}\vartheta(x)})$$
$$=\frac{1}{\pi\mathrm{i}}\oint_C \mathrm{d}z\ln z=0. \tag{A.30}$$

The r.h.s. of (A.30) vanishes due to Cauchy's theorem which allows one the contraction of the contour C to a contour of zero length. This leads again to the chiral Jacobian (A.26).

Appendix B. Stability of chirally broken phase under $\tilde\rho$ field fluctuations

Here we discuss the stability of the chirally broken phase under $\tilde\rho$ field fluctuations. For this aim we calculate the effective Lagrangian of the $\tilde\rho$ field and demonstrate the decoupling of the $\tilde\rho$ field.

The evaluation of the effective Lagrangian of the $\tilde\rho$ field runs in the following way. First, we rewrite the functional determinant (3.5)

$$\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu-\sigma-\mathrm{i}\gamma^5\varphi)$$
$$=\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu-\rho\mathrm{e}^{\mathrm{i}\vartheta})$$
$$=\mathrm{Det}(\mathrm{e}^{\mathrm{i}\gamma^5\vartheta/2}(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu A_\mu-\rho)\mathrm{e}^{\mathrm{i}\gamma^5\vartheta/2})$$
$$=\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu A_\mu-\rho), \tag{B.1}$$

where A_μ is given by (3.31)

$$A_\mu(x)=\frac{1}{2}\varepsilon_{\mu\nu}\partial^\nu\vartheta(x). \tag{B.2}$$

In (B.1) we have used the fact that the Jacobian of the chiral rotation is equal to unity, see (A.26).

Secondly, we make a shift $\rho=M+\tilde\rho$ and represent the functional determinant in the r.h.s. of (B.1) as follows:

$$\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu+\gamma^\mu A_\mu-\rho)$$
$$=\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu-M+\gamma^\mu A_\mu)$$
$$\times\mathrm{Det}\left(1-\frac{1}{\mathrm{i}\gamma^\mu\partial_\mu-M+\gamma^\mu A_\mu}\tilde\rho\right). \tag{B.3}$$

The determinant $\mathrm{Det}(\mathrm{i}\gamma^\mu\partial_\mu-M+\gamma^\mu A_\mu)$ describes the effective Lagrangian of the ϑ field that is given by (3.35). It is convenient to recast the determinant containing the $\tilde\rho$ field into the form

$$\mathrm{Det}\left(1-\frac{1}{\mathrm{i}\gamma^\mu\partial_\mu-M+\gamma^\mu A_\mu}\tilde\rho\right)$$

$$= \mathrm{Det}\Big(1 - \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\tilde{\rho} + \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\gamma^\nu A_\nu \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\tilde{\rho} - \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\gamma^\alpha A_\alpha \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\gamma^\beta A_\beta \times \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\tilde{\rho} + \ldots\Big). \qquad (B.4)$$

It is obvious that even if the scale of the $\tilde{\rho}$ field is of order $O(M)$, the contribution of the ϑ field should be of order $O(\partial_\mu\vartheta/M)$. This implies that the ϑ field is decoupled from the $\tilde{\rho}$ field and (B.4). This allows one to consider the approximate form for the determinant (B.4)

$$\mathrm{Det}\left(1 - \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M + \gamma^\mu A_\mu}\tilde{\rho}\right) \simeq \mathrm{Det}\left(1 - \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\tilde{\rho}\right).$$

The effective Lagrangian of the $\tilde{\rho}$ field is then given by

$$\mathcal{L}_{\rm eff}[\tilde{\rho}(x)] = -\mathrm{itr}\left\langle x\left|\ln\left(1 - \frac{1}{\mathrm{i}\gamma^\mu\partial_\mu - M}\tilde{\rho}\right)\right|x\right\rangle - \frac{M}{2g}\tilde{\rho}(x) - \frac{1}{2g}\tilde{\rho}^2(x) - \mathrm{i}\delta^{(2)}(0)\ln\left(1 + \frac{\tilde{\rho}(x)}{M}\right). \qquad (B.5)$$

The last term comes from the measure of the path integral in the polar representation [45]: $\mathcal{D}\sigma\mathcal{D}\varphi = \mathrm{Det}\rho\mathcal{D}\rho\mathcal{D}\vartheta = \exp\{\delta^{(2)}(0)\int \mathrm{d}^2x\ln\rho(x)\}\mathcal{D}\rho\mathcal{D}\vartheta = \exp\{\delta^{(2)}(0)\int \mathrm{d}^2x\ln\big(M + \tilde{\rho}(x)\big)\}\mathcal{D}\tilde{\rho}\mathcal{D}\vartheta$. This adds a contact term $-\mathrm{i}\delta^{(2)}(0)\ln(1 + \tilde{\rho}(x)/M)$, which serves to cancel divergences appearing from the loop contributions of the $\tilde{\rho}$ field [45,46].

It is obvious that the effective potential of the $\tilde{\rho}$ field independent of gradients $\partial_\alpha\tilde{\rho}(x)$ is completely defined by the effective potential (3.21) and can be written as

$$V[\tilde{\rho}(x)] = \frac{1}{4\pi}\Big[(M^2 + 2M\tilde{\rho}(x) + \tilde{\rho}^2(x)) \times \ln\left(1 + 2\frac{\tilde{\rho}(x)}{M} + \frac{\tilde{\rho}^2(x)}{M^2}\right) - (\Lambda^2 + M^2 + 2M\tilde{\rho}(x) + \tilde{\rho}^2(x)) \times \ln\left(1 + 2\frac{M\tilde{\rho}(x)}{\Lambda^2 + M^2} + \frac{\tilde{\rho}^2(x)}{\Lambda^2 + M^2}\right)\Big], \qquad (B.6)$$

where we have used (3.22).

In order to understand what kind of rescaling of the $\tilde{\rho}$ field should be carried out it is sufficient to calculate a two-vertex diagram contribution keeping only the contribution of the gradient $\partial_\mu\tilde{\rho}(x)$. It reads

$$\mathcal{L}^{(2)}_{\rm eff}[\partial_\alpha\tilde{\rho}(x)] = \frac{1}{8\pi}\partial_\alpha\tilde{\rho}(x)\int_0^1 \mathrm{d}\alpha\left[\frac{1}{M^2 + \alpha(1-\alpha)\Box} - \frac{1}{\Lambda^2 + M^2 + \alpha(1-\alpha)\Box}\right]\partial^\alpha\tilde{\rho}(x). \qquad (B.7)$$

Expanding the integrand in powers of $1/M^2$ and $1/(\Lambda^2 + M^2)$ we obtain

$$\mathcal{L}^{(2)}_{\rm eff}[\partial_\alpha\tilde{\rho}(x)] = \frac{1}{8\pi}\frac{\Lambda^2}{M^2(\Lambda^2 + M^2)}\partial_\alpha\tilde{\rho}(x)\partial^\alpha\tilde{\rho}(x) + \frac{1}{48\pi}\frac{\Lambda^2(\Lambda^2 + 2M^2)}{M^4(\Lambda^2 + M^2)^2} \times \partial_\alpha\partial_\beta\tilde{\rho}(x)\partial^\alpha\partial^\beta\tilde{\rho}(x) + \ldots \qquad (B.8)$$

In order to get a correct kinetic term we have to rescale the $\tilde{\rho}$ field

$$\tilde{\rho}(x) = \sqrt{4\pi M^2\left(1 + \frac{M^2}{\Lambda^2}\right)}\,v(x). \qquad (B.9)$$

In terms of the v field the Lagrangian (B.8) reads

$$\mathcal{L}^{(2)}_{\rm eff}[\partial_\alpha v(x)] = \frac{1}{2}\partial_\alpha v(x)\partial^\alpha v(x) + \frac{1}{3}\frac{1 + \dfrac{\Lambda^2}{2M^2}}{1 + \dfrac{\Lambda^2}{2M^2}}\frac{1}{M^2} \times \partial_\alpha\partial_\beta v(x)\partial^\alpha\partial^\beta v(x) + \ldots \qquad (B.10)$$

Hence, higher gradients of the v field would enter in the form of the ratios $O(\partial_\alpha/M)$ and can be dropped at leading order in the $1/M$ expansion.

Thus, the effective Lagrangian of the rescaled $\tilde{\rho}$ field, the v field, is defined by

$$\mathcal{L}_{\rm eff}[v(x)] = \frac{1}{2}\partial_\alpha v(x)\partial^\alpha v(x) - \frac{1}{2}M_v^2 N[v(x)], \qquad (B.11)$$

where $M_v = 2M$ is the mass of the v field. This agrees with the classical Nambu–Jona–Lasinio model [19–22], where the mass of the σ meson is twice the mass of the dynamical fermions. Then, the functional $N[v(x)]$ is equal to

$$N[v(x)] \qquad (B.12)$$

$$= \frac{1}{8\pi}\Bigg\{\Big[1 + \sqrt{16\pi\left(1 + \frac{M^2}{\Lambda^2}\right)}v(x) + 4\pi\left(1 + \frac{M^2}{\Lambda^2}\right)v^2(x)\Big]\ln\Big[1 + \sqrt{16\pi\left(1 + \frac{M^2}{\Lambda^2}\right)}v(x) + 4\pi\left(1 + \frac{M^2}{\Lambda^2}\right)v^2(x)\Big] - \left(1 + \frac{\Lambda^2}{M^2}\right) \times \left[1 + \frac{M^2}{\Lambda^2}\sqrt{\frac{16\pi\Lambda^2}{\Lambda^2 + M^2}}v(x) + 4\pi\frac{M^2}{\Lambda^2}v^2(x)\right] \times \ln\left[1 + \frac{M^2}{\Lambda^2}\sqrt{\frac{16\pi\Lambda^2}{\Lambda^2 + M^2}}v(x) + 4\pi\frac{M^2}{\Lambda^2}v^2(x)\right]\Bigg\}.$$

Expanding the functional $N[v(x)]$ in powers of $v(x)$ around $v(x) = 0$ we obtain

$$N[v(x)] = v^2(x) + O(v^3(x)). \qquad (B.13)$$

Since for the derivation of effective Lagrangians the ratio M^2/Λ^2 is fixed at $\Lambda \to \infty$, the functional $\exp\left\{-(\mathrm{i}/2)M_v^2\int \mathrm{d}^2x N[v(x)]\right\}$ reduces in the $\Lambda \to \infty$ limit to the functional δ function $\delta[v(x)]$:

$$\exp\left\{-\frac{\mathrm{i}}{2}M_v^2\int \mathrm{d}^2x N[v(x)]\right\} \stackrel{\Lambda\to\infty}{\longrightarrow} \prod_x \delta[v(x)]. \quad \text{(B.14)}$$

Thus, the generating functional of Green functions of the v field

$$\begin{aligned} Z[q] = \int \mathcal{D}v \exp\Bigg\{\frac{\mathrm{i}}{2}\int \mathrm{d}^2x \\ \times\Bigg[\partial_\alpha v(x)\partial^\alpha v(x) - M_v^2 N[v(x)] \\ -\mathrm{i}\delta^{(2)}(0)\ln\left(1+\sqrt{4\pi\left(1+\frac{M^2}{\Lambda^2}\right)v(x)}\right)\Bigg] \\ +\mathrm{i}\int \mathrm{d}^2x q(x)v(x)\Bigg\} \end{aligned} \quad \text{(B.15)}$$

reduces in the $\Lambda \to \infty$ limit to the form

$$Z[q] = \int \mathcal{D}v\delta[v]\exp\left\{\frac{\mathrm{i}}{2}\int \mathrm{d}^2x\partial_\alpha v(x)\partial^\alpha v(x)\right\}. \quad \text{(B.16)}$$

The appearance of the δ functional, $\delta[v]$, is evidence that the classical value of the v field is zero, $v_{\mathrm{cl}}(x) = 0$.

The generating functional (B.16) does not depend on the external source $q(x)$. This confirms the decoupling of the v field that corresponds to the decoupling of the $\tilde{\rho}$ field. Thereby, no contributions can appear due to fluctuations of the ρ field around the minimum of the effective potential (3.21). This implies the stability of the chirally broken phase and the non-perturbative vacuum described by the effective potential (3.21) under fluctuations of the ρ field. We would like to accentuate that the decoupling of the $\tilde{\rho}$ field agrees fully with the decoupling theorem derived by Appelquist and Carazzone [41].

Appendix C. Solutions of equations of motion (6.45) and (6.46) for the ansatz (6.55)

It is easy to show that for the ansatz (6.55) the equations of motion (6.45) reduce to the form

$$\begin{aligned} -\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\xi(x,t) &= M\mathrm{e}^{-\omega}, \\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\eta(x,t) &= M\mathrm{e}^{+\omega}, \end{aligned} \quad \text{(C.1)}$$

whereas the equations of motion become split into a set of first order differential equations

$$\begin{aligned} -\frac{\partial}{\partial t}\xi(x,t) &= \frac{M}{g}\left(+\frac{a-b}{2}\mathrm{e}^{+\omega}+\frac{a+b}{2}\mathrm{e}^{-\omega}\right), \\ -\frac{\partial}{\partial x}\xi(x,t) &= \frac{M}{g}\left(-\frac{a-b}{2}\mathrm{e}^{+\omega}+\frac{a+b}{2}\mathrm{e}^{-\omega}\right), \\ \frac{\partial}{\partial t}\eta(x,t) &= \frac{M}{g}\left(+\frac{a+b}{2}\mathrm{e}^{+\omega}+\frac{a-b}{2}\mathrm{e}^{-\omega}\right), \\ \frac{\partial}{\partial x}\eta(x,t) &= \frac{M}{g}\left(-\frac{a+b}{2}\mathrm{e}^{+\omega}+\frac{a-b}{2}\mathrm{e}^{-\omega}\right). \end{aligned} \quad \text{(C.2)}$$

Due to the relation $a+b=g$ the equations of (C.1) are consistent with (C.2). Using the relations $a+b=g$ and $a-b=1/c$ (C.1) and (C.2) can be rewritten in the equivalent form

$$\begin{aligned} \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\xi(x,t) &= -\frac{M}{gc}\mathrm{e}^{+\omega}, \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\xi(x,t) &= -M\mathrm{e}^{-\omega} \end{aligned} \quad \text{(C.3)}$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\eta(x,t) &= +M\mathrm{e}^{+\omega}, \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\eta(x,t) &= -\frac{M}{gc}\mathrm{e}^{-\omega}. \end{aligned}$$

The solutions of the differential equations (C.4) and (C.4) read

$$\begin{aligned} \xi(x,t) &= \xi_0 - \frac{M}{gc}\mathrm{e}^{+\omega}(t-x) - M\mathrm{e}^{-\omega}(t+x), \\ \eta(x,t) &= \eta_0 + M\mathrm{e}^{+\omega}(t-x) - \frac{M}{gc}\mathrm{e}^{-\omega}(t+x), \end{aligned} \quad \text{(C.4)}$$

where ξ_0 and η_0 are integration constants and c is the Schwinger term (6.66).

For $\vartheta(x,t)$ we obtain

$$\begin{aligned} \vartheta(x,t) &= \frac{1}{\beta}[\xi(x,t)+\eta(x,t)] \\ &= \frac{\xi_0+\eta_0}{\beta} + \frac{M}{\beta}\left(1-\frac{1}{gc}\right)\mathrm{e}^{+\omega}(t-x) \\ &\quad -\frac{M}{\beta}\left(1+\frac{1}{gc}\right)\mathrm{e}^{-\omega}(t+x), \end{aligned} \quad \text{(C.5)}$$

where β is defined by (4.5).

Thus we have confirmed the consistency of the equations of motion (6.45) and (6.46) and their consistency with the ansatz (6.55).

Notice that the factors $\mathrm{e}^{\pm\omega}$ can be removed by an appropriate Lorentz boost. This yields

$$\begin{aligned} \xi(x,t) &= \xi_0 - \frac{M}{gc}(t-x) - M(t+x) \\ &= \xi_0 - M\left(1+\frac{1}{gc}\right)t - M\left(1-\frac{1}{gc}\right)x, \\ \eta(x,t) &= \eta_0 + M(t-x) - \frac{M}{gc}(t+x) \\ &= \eta_0 + M\left(1-\frac{1}{gc}\right)t - M\left(1+\frac{1}{gc}\right)x, \end{aligned}$$

$$\begin{aligned}\vartheta(x,t) &= \frac{\xi_0+\eta_0}{\beta} + \frac{M}{\beta}\left(1-\frac{1}{gc}\right)(t-x)\\ &\quad -\frac{M}{\beta}\left(1+\frac{1}{gc}\right)(t+x)\\ &= \frac{\xi_0+\eta_0}{\beta} - \frac{2M}{\beta gc}t - \frac{2M}{\beta}x. \end{aligned} \tag{C.6}$$

This simplifies the solutions following from the ansatz (6.55) describing a helical wave as discussed in [1].

Appendix D. Quantum field theory of free massive and massless fermion fields in 1 + 1-dimensional space-time

The main aim of this appendix is to specify the definitions of massive and massless fermion fields in 1+1-dimensional space-time.

Let $\psi(x)$ be a free massive fermion field with mass m obeying the Dirac equation of motion

$$(\mathrm{i}\gamma_\mu\partial^\mu - m)\psi(x) = 0. \tag{D.1}$$

The quantization of field $\psi(x)$ goes via a solution of (D.1) in terms of plane waves:

$$\begin{aligned}\psi(x) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{\sqrt{2\pi}}\frac{1}{\sqrt{2p^0}}\\ &\quad\times\left[u(p^0,p^1)a(p^1)\mathrm{e}^{-\mathrm{i}p\cdot x} + v(p^0,p^1)b^\dagger(p^1)\mathrm{e}^{\mathrm{i}p\cdot x}\right],\\ \bar\psi(x) &= \psi^\dagger(x)\gamma^0\\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{\sqrt{2\pi}}\frac{1}{\sqrt{2p^0}}\left[\bar u(p^0,p^1)a^\dagger(p^1)\mathrm{e}^{\mathrm{i}p\cdot x}\right.\\ &\quad\left.+\bar v(p^0,p^1)b(p^1)\mathrm{e}^{-\mathrm{i}p\cdot x}\right], \end{aligned}\tag{D.2}$$

where $p\cdot x = p^0x^0 - \mathrm{i}p^1x^1$. The creation $a^\dagger(p^1)(b^\dagger(p^1))$ and annihilation $a(p^1)(b(p^1))$ operators of fermions (anti-fermions) with momentum p^1 and energy $p^0 = ((p^1)^2 + m^2)^{1/2}$ obey the anti-commutation relations

$$\begin{aligned}\{a(p^1),a^\dagger(q^1)\} &= \{b(p^1),b^\dagger(q^1)\} = \delta(p^1-q^1),\\ \{a(p^1),a(q^1)\} &= \{a^\dagger(p^1),a^\dagger(q^1)\} = \{b(p^1),b(q^1)\}\\ &= \{b^\dagger(p^1),b^\dagger(q^1)\} = 0. \end{aligned}\tag{D.3}$$

The wave functions $u(p^0,p^1)$ and $v(p^0,p^1) = u(-p^0,-p^1)$ are the solutions of the Dirac equation in the momentum representation for positive and negative energies, respectively. They are defined by

$$\begin{aligned}u(p^0,p^1) &= \begin{pmatrix}\sqrt{p^0+p^1}\\ \sqrt{p^0-p^1}\end{pmatrix},\\ \bar u(p^0,p^1) &= (\sqrt{p^0-p^1},\sqrt{p^0+p^1}),\\ v(p^0,p^1) &= \begin{pmatrix}\sqrt{p^0+p^1}\\ -\sqrt{p^0-p^1}\end{pmatrix},\\ \bar v(p^0,p^1) &= (-\sqrt{p^0-p^1},\sqrt{p^0+p^1}) \end{aligned}\tag{D.4}$$

at $p^0 = ((p^1)^2 + m^2)^{1/2}$ and normalized to

$$\begin{aligned}u^\dagger(p^0,p^1)u(p^0,p^1) &= v^\dagger(p^0,p^1)v(p^0,p^1) = 2p^0,\\ \bar u(p^0,p^1)u(p^0,p^1) &= -\bar v(p^0,p^1)v(p^0,p^1) = 2m,\\ \bar u(p^0,p^1)v(p^0,p^1) &= \bar v(p^0,p^1)u(p^0,p^1) = 0. \end{aligned}\tag{D.5}$$

The functions $u(p^0,p^1)$ and $v(p^0,p^1)$ satisfy the following matrix relations:

$$\begin{aligned}&u(p^0,p^1)\bar u(p^0,p^1)\\ &= \begin{pmatrix}\sqrt{p^0+p^1}\\ \sqrt{p^0-p^1}\end{pmatrix}(\sqrt{p^0-p^1},\sqrt{p^0+p^1})\\ &= \begin{pmatrix}\sqrt{(p^0)^2-(p^1)^2} & p^0+p^1\\ p^0-p^1 & \sqrt{(p^0)^2-(p^1)^2}\end{pmatrix}\\ &= \begin{pmatrix}m & p^0+p^1\\ p^0-p^1 & m\end{pmatrix}\\ &= \gamma^0p^0 - \gamma^1p^1 + m = \hat p + m,\\ &v(p^0,p^1)\bar v(p^0,p^1)\\ &= \begin{pmatrix}\sqrt{p^0+p^1}\\ -\sqrt{p^0-p^1}\end{pmatrix}(-\sqrt{p^0-p^1},\sqrt{p^0+p^1})\\ &= \begin{pmatrix}-\sqrt{(p^0)^2-(p^1)^2} & p^0+p^1\\ p^0-p^1 & -\sqrt{(p^0)^2-(p^1)^2}\end{pmatrix}\\ &= \begin{pmatrix}-m & p^0+p^1\\ p^0-p^1 & -m\end{pmatrix}\\ &= \gamma^0p^0 - \gamma^1p^1 - m = \hat p - m. \end{aligned}\tag{D.6}$$

The causal Green function of a free massive fermion field $S_{\mathrm{F}}(x-y)$ is defined by

$$\begin{aligned}S_{\mathrm{F}}(x-y) &= \mathrm{i}\langle 0|T(\psi(x)\bar\psi(y))|0\rangle\\ &= \mathrm{i}\theta(x^0-y^0)\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1+m}{2p^0}\\ &\quad\times\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)+\mathrm{i}p^1(x^1-y^1)}\\ &\quad-\mathrm{i}\theta(y^0-x^0)\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1-m}{2p^0}\\ &\quad\times\mathrm{e}^{-\mathrm{i}p^0(y^0-x^0)+\mathrm{i}p^1(y^1-x^1)}, \end{aligned}\tag{D.7}$$

where $\theta(z^0)$ is the Heaviside function.

Using the integral representation for the Heaviside function [47]

$$\theta(z^0) = \int_{-\infty}^{\infty}\frac{\mathrm{d}q^0}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}q^0z^0}}{q^0-\mathrm{i}0} \tag{D.8}$$

we recast the r.h.s. of (D.7) into the form

$$\begin{aligned}S_{\mathrm{F}}(x-y) &= \mathrm{i}\langle 0|T(\psi(x)\bar\psi(y))|0\rangle\\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q^0}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1+m}{2p^0(q^0-\mathrm{i}0)}\\ &\quad\times\mathrm{e}^{\mathrm{i}(q^0-p^0)(x^0-y^0)+\mathrm{i}p^1(x^1-y^1)}\end{aligned}$$

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q^0}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1-m}{2p^0(q^0-\mathrm{i}0)}\\
&\times\mathrm{e}^{-\mathrm{i}(q^0-p^0)(x^0-y^0)-\mathrm{i}p^1(x^1-y^1)}\\
&=\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q^0}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1+m}{2p^0(p^0-q^0-\mathrm{i}0)}\\
&\times\mathrm{e}^{-\mathrm{i}q^0(x^0-y^0)+\mathrm{i}p^1(x^1-y^1)}\\
&-\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q^0}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1-m}{2p^0(p^0+q^0-\mathrm{i}0)}\\
&\times\mathrm{e}^{-\mathrm{i}q^0(x^0-y^0)-\mathrm{i}p^1(x^1-y^1)}\\
&=\int_{-\infty}^{\infty}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{m+\hat{p}}{m^2-p^2-\mathrm{i}0}\\
&\times\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}. \qquad (\mathrm{D}.9)
\end{aligned}
$$

A direct calculation of the integral over p yields

$$
\begin{aligned}
S_{\mathrm{F}}(x-y)&=\frac{m}{2\pi}\frac{-(\hat{x}-\hat{y})}{\sqrt{-(x-y)^2}}K_1\left(m\sqrt{-(x-y)^2}\right)\\
&+\mathrm{i}\frac{m}{2\pi}K_0\left(m\sqrt{-(x-y)^2}\right). \qquad (\mathrm{D}.10)
\end{aligned}
$$

For the product $(-\mathrm{i})\gamma_\mu S_{\mathrm{F}}(x-y)\gamma^\mu$ we get

$$
(-\mathrm{i})\gamma_\mu S_{\mathrm{F}}(x-y)\gamma^\mu=\frac{m}{2\pi}K_0\left(m\sqrt{-(x-y)^2}\right). \qquad (\mathrm{D}.11)
$$

At $(x-y)\to 0$ this agrees with our calculation of $\gamma_\mu\langle 0|\psi(x,t)\bar{\psi}(x,t)|0\rangle\gamma^\mu$ given by (6.14) and (6.15).

The solution of a massless fermion field can be obtained from the solution (D.2) in the limit $m\to 0$. The functions $u(p^0,p^1)$ and $v(p^0,p^1)$ are defined by (D.4) at $p^0=|p^1|$.

We would like to emphasize that our solution for a free massless fermion field has a phase convention different from that used by Thirring [3] and Klaiber [5] who set

$$
\begin{aligned}
u(p_0,p)&=v(p_0,p)\\
&=\begin{pmatrix}\sqrt{p_0+p}\\ \sqrt{p_0-p}\end{pmatrix}=\sqrt{2p_0}\begin{pmatrix}\theta(+p)\\ \theta(-p)\end{pmatrix}, \qquad (\mathrm{D}.12)
\end{aligned}
$$

where $\theta(\pm p)$ are Heaviside functions.

The causal Green function $S_{\mathrm{F}}(x-y)$ of a free massless fermion field is defined by

$$
\begin{aligned}
S_{\mathrm{F}}(x-y)&=\mathrm{i}\langle 0|T(\psi(x)\bar{\psi}(y)|0\rangle\\
&=\mathrm{i}\theta(x^0-y^0)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle\\
&-\mathrm{i}\theta(y^0-x^0)\langle 0|\bar{\psi}^Ty)\psi^T(x)|0\rangle\\
&=\mathrm{i}\theta(x^0-y^0)\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\frac{1}{2p^0}[u(p^0,p^1)\bar{u}(p^0,p^1)]\\
&\times\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)+\mathrm{i}p^1(x^1-y^1)}\\
&-\mathrm{i}\theta(y^0-x^0)\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\frac{1}{2p^0}[v(p^0,p^1)\bar{v}(p^0,p^1)]\\
&\times\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)+\mathrm{i}p^1(x^1-y^1)}\\
&=\mathrm{i}\theta(x^0-y^0)\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1}{2p^0}\\
&\times\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)+\mathrm{i}p^1(x^1-y^1)}\\
&-\mathrm{i}\theta(y^0-x^0)\int_{-\infty}^{\infty}\frac{\mathrm{d}p^1}{2\pi}\frac{\gamma^0p^0-\gamma^1p^1}{2p^0}\\
&\times\mathrm{e}^{-\mathrm{i}p^0(y^0-x^0)+\mathrm{i}p^1(y^1-x^1)}\\
&=\mathrm{i}\varepsilon(x^0-y^0)(\hat{x}-\hat{y})\delta((x-y)^2)\\
&+\frac{1}{2\pi}\frac{\hat{x}-\hat{y}}{(x-y)^2}\\
&=\frac{1}{2\pi}\frac{\hat{x}-\hat{y}}{(x-y)^2-\mathrm{i}0\cdot\varepsilon(x^0-y^0)}, \qquad (\mathrm{D}.13)
\end{aligned}
$$

where $\varepsilon(x^0-y^0)$ is a sign function. This is a well-known expression for the causal Green function of a free massless fermion field in 1+1-dimensional space-time [48] (see also [47]).

References

1. M. Remoissenet, in Waves called solitons, concepts and experiments (Springer-Verlag, Heidelberg 1994)
2. W. Thirring, Ann. Phys. (N.Y.) **3**, 91 (1958)
3. S. Coleman, Phys. Rev. D **11**, 2088 (1975)
4. T.H.R. Skyrme, Proc. Roy. Soc. A **247**, 260 (1958); ibid. A **262**, 237 (1961)
5. B. Klaiber, in Lectures in theoretical physics, Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1967, edited by A. Barut, W. Brittin (Gordon and Breach, New York 1968), Vol. X, part A, pp. 141–176
6. K. Furuya, Re.E. Gamboa Saravi, F.A. Schaposnik, Nucl. Phys. B **208**, 159 (1982)
7. J. Fröhlich, P. Marchetti, Commun. Math. Phys. **116**, 127 (1988)
8. S. Mandelstam, Phys. Rev. D **11**, 3026 (1975)
9. P.H. Damgaard, H.B. Nielsen, R. Sollacher, Nucl. Phys. B **385**, 227 (1992) and references therein
10. J. Thomassen, PhD, TUWien, 15 January 2000
11. J. Alfaro, P.H. Damgaard, Ann. of Phys. (N.Y.) **202**, 398 (1990)
12. K. Fujikawa, Phys. Rev. Lett. **44**, 1195 (1979); Phys. Rev. D **21**, 2848 (1982); Phys. Rev. D **22**, 1499(E) (1980)
13. R. Roskies, F.A. Schaposnik, Phys. Rev. D **23**, 558 (1981); R.E. Gamboa Saravi, F.A. Schaposnik, J.E. Solomin, Nucl. Phys. B **153**, 112 (1979); R.E. Gamboa Savari, M.A. Muschetti, F.A. Schaposnik, J. E. Solomin, Ann. of Phys. (N.Y.) **157**, 360 (1984)
14. O. Alvarez, Nucl. Phys. B **238**, 61 (1984)
15. S.K. Hu, B.L. Young, D.W. McKay, Phys. Rev. D **30**, 836 (1984), C.M. Naón, Phys. Rev. D **31**, 2035 (1985)
16. H. Dorn, Phys. Lett. B **167**, 86 (1986)
17. R.A. Bertlmann, in Anomalies in quantum field theory, Oxford Science Publications (Clarendon Press, Oxford 1996), pp. 248–286
18. Y. Nambu, G. Jona–Lasinio, Phys. Rev. **122**, 345 (1961); ibid. **124**, 246 (1961)
19. K. Kikkawa, Progr. Theor. Phys. **56**, 947 (1976); H. Kleinert, Proceedings of International Summer School of Subnuclear Physics, Erice 1976, edited by A. Zichichi, p. 289; A. Dhar, R. Shankar, S.R. Wadia, Phys. Rev. D **31**, 3256 (1985); D. Ebert, H. Reinhardt, Nucl. Phys. B **271**, 188 (1986); M. Wakamatsu, Ann. of Phys. (N.Y.) **193**, 287 (1989)

20. S.P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992); T. Hatsuda, T. Kunihiro, Phys. Rep. **247**, 221 (1994)
21. A.N. Ivanov, M. Nagy, N.I. Troitskaya, Int. J. Mod. Phys. A **7**, 7305 (1992); A.N. Ivanov, Int. J. Mod. Phys. A **8**, 853 (1993)
22. M. Faber, A.N. Ivanov, W. Kainz, N.I. Troitskaya, Z. Phys. C **74**, 721 (1997); Phys. Lett. B **386**, 198 (1996); M. Faber, A.N. Ivanov, A. Müller, N.I. Troitskaya, M. Zach, Eur. Phys. J. C **7**, 685 (1999); ibid. C **10**, 537 (1999)
23. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **106**, 162 (1957); Phys. Rev. **108**, 1175 (1957)
24. A.N. Ivanov, Nonperturbative phenomena in quantum chromodynamics, Lectures delivered at Institute of Nuclear Physics in Vienna University of Technology in Winter semester in 1995, Part 2
25. N.D. Mermin, H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966)
26. S. Coleman, Comm. Math. Phys. **31**, 259 (1973)
27. C. Itzykson, J.-B. Zuber, in Quantum field theory (McGraw–Hill Book Company, New York 1980), p. 525
28. K. Huang, in Quantum field theory, From operators to path integrals (John Wiley and Sons, Inc., New York 1998), pp. 363–367
29. E.M. Lifshitz, L.P. Pitaevskii, in Relativistic quantum theory, part 2 (Pergamon Press, New York 1974), p. 503; (see also [17], p. 136)
30. C.G. Callan, R.F. Dashen, D.H. Sharp, Phys. Rev. **165**, 1883 (1968)
31. H. Sugawara, Phys. Rev. **170**, 1659 (1968)
32. C.M. Sommerfield, Phys. Rev. **176**, 2019 (1968); H. Georgi, Phys. Rev. D **2**, 2908 (1970)
33. J. Schwinger, Phys. Rev. Lett. **3**, 296 (1959)
34. C.M. Sommerfield, Ann. of Phys. (N.Y.) **26**, 1 (1963)
35. D. Finkelstein, J. Math. Phys. **7**, 1218 (1966); L.D. Faddeev, V.E. Korepin, Phys. Rep. C **42**, 1 (1978) (see also [28], pp. 380–388)
36. M. Gell–Mann, R.J. Oakes, B. Renner, Phys. Rev. **175**, 2195 (1968)
37. P.C. Hohenberg, Phys. Rev. **158**, 383 (1967)
38. P.B. Vigman, A.I. Larkin, Zh. Exper. Theor. Fiz. **72**, 857 (1977) (in Russian)
39. J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. **127**, 965 (1962)
40. See [27], pp. 519–521
41. T. Appelquist, J. Carazzone, Phys. D **11**, 2856 (1975)
42. M. Faber, Few Body System **30**, 149 (2001)
43. K. Göke, J.N. Urbano, M. Fiolhais, M. Harvey, Phys. Lett. B **164**, 249 (1985)
44. See [17], pp. 284–286
45. D.J. Gross, R. Jackiw, Phys. Rev. D **6**, 477 (1972); L. Dolan, R. Jackiw, Phys. Rev. D **9**, 2904 (1974)
46. A. Salam, J. Strathdee, Phys. Rev. D **2**, 2869 (1970); J. Honerkamp, K. Meetz, Phys. Rev. D **3**, 1996 (1971); J.M. Charap, Phys. Rev. D **3**, 1998 (1971)
47. N.N. Bogoliubov, D.V. Shirkov, in Introduction to the theory of quantized fields (Interscience Publishers, Inc., New York 1959), pp. 136–153; p. 648
48. B. Klaiber, Helv. Phys. Acta **37**, 554 (1964)